

# Efficient Public Investment in a Model with Transition Dynamics

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Pierre-Daniel G. Sarte and Jorge Soares

**A** long history of debate exists among macroeconomists regarding the degree to which public capital contributes to overall economic activity. Estimates of the output elasticity with respect to public capital range anywhere from 0.06 in early work by Ratner (1983) to as high as 0.39 in a widely cited study by Aschauer (1989). This elasticity refers to the percentage change in GDP induced by a given percentage change in government capital.<sup>1</sup> Glomm and Ravikumar (1997, 197) remark that economists have generally been skeptical of Aschauer's estimate, mainly because "the productivity of public capital is simply not believed to be larger than the productivity of the private capital stock (which is roughly 0.36)." That being said, few would argue that public infrastructure plays no productive role, and estimates of the public capital elasticity of output lying between 0.05 and 0.15 are often put forward.<sup>2</sup>

In an economy where government investment in equipment and structures complements private investment, it is only natural to ask what factors determine the optimal share of public investment in output. The trade-off involves public capital that is productive but that must also be financed through distortionary taxation.

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■ Jorge Soares is assistant professor in the Department of Economics at George Washington University, 2201 G Street NW, Washington, DC 20052.

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<sup>1</sup> For any variables  $x$  and  $y$ , the elasticity of  $y$  with respect to  $x$  is defined as  $\varepsilon \equiv \left(\frac{dy}{dx}\right) \left(\frac{x}{y}\right)$ . Thus, if  $y = x^\theta$ ,  $\varepsilon = \theta$ .

<sup>2</sup> See Glomm and Ravikumar (1997) for a survey of recent estimates.

The literature has shown that the optimal share of gross public investment should equal the government capital elasticity of output. However, most influential papers in this literature utilize a very special class of growth models: endogenous growth models without transition dynamics.<sup>3</sup> In these models, the elasticities of output with respect to private and public capital must add up to one. This restriction is empirically implausible because the private capital elasticity of output is approximately 0.36 in the U.S. economy. Thus, if this restriction were true, then the optimal share of public investment in GDP would be 0.64.

In this article, we show that this implausible restriction is an implication of the lack of transition dynamics. We revisit the optimal choice of public investment in a more general and plausible model that allows for gradual transitions between steady states. Since endogenous growth is not essential to our argument, we revert to the more conventional growth model with exogenous technical progress.

Contrary to previous work, we show that (i) the optimal share of gross public investment in output should be less than its elasticity along a balanced growth path; (ii) this share depends in important ways on assumed preferences and technology, including the underlying rate of productivity growth; (iii) the optimal sequence of public investments is not time invariant, and furthermore, a policy aimed at implementing it is not time consistent; and (iv) the government capital elasticity of output is likely to be relatively low at less than 0.1 if the observed U.S. ratios of gross public investment to output and of public capital to private capital are approximately optimal.

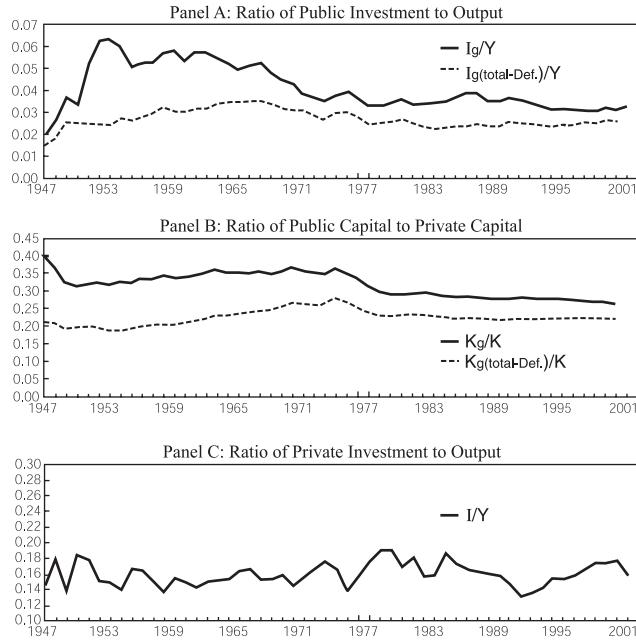
This article is organized as follows: Section 1 gives an overview of public investment in U.S. data. Section 2 sets the basic theoretical framework and derives the optimal steady state share of public investment in output. In contrast, Section 3 describes the solution to the full optimal policy problem with commitment. Section 4 offers concluding remarks.

## 1. PUBLIC INVESTMENT IN U.S. DATA

Panel A in Figure 1 depicts the behavior of U.S. gross public investment relative to GDP during the postwar period. Observe first that the degree of public investment in the United States is nonnegligible and roughly comparable as a percentage of GDP to that of its net exports. Public investment has amounted to as much as 6 percent of GDP at its peak in the 1950s, but it has also been relatively constant. The run-up in public investment apparent during the 1950s and early 1960s captures a discrete increase in military spending related to the Korean War, as well as increases in spending on schools and

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<sup>3</sup> See Barro (1990), Glomm and Ravikumar (1997), and Aschauer (2000), among many others.

**Figure 1** Public Investment and Public Capital over the Postwar Period

highways. Outside of changes in military spending over time, the share of public investment in output has stayed mostly flat at approximately 3 percent.

Panel B in Figure 1 shows the ratio of public to private capital from 1947 to today. Abstracting from public capital tied to national defense, which includes aircrafts, ships, vehicles, electronic equipment, and missiles, this ratio has never shown much variation, moving only between 0.20 and 0.25 during the entire period. We see a slight run-up in public capital during the mid-1960s, which corresponds to the construction of the interstate highway system. Interestingly, Fernald (1999) suggests that this construction provided a significant onetime increase in productivity. Note in Panel C in Figure 1 that private investment relative to output has always been much larger than public investment, averaging around 15 percent since World War II.

The fact that some degree of public investment has consistently taken place over the years suggests that the provision of infrastructure such as highways, airports, and even public sector research and development indeed contributes to economic activity. Thus, we will now study a simple economic environment where public capital plays a productive role and examine the factors that determine efficient public investment.

## 2. THEORETICAL FRAMEWORK

Consider a closed economy in which a large number of firms produce a single final good according to the technology:

$$Y_t = K_t^\alpha (z_t l_t)^{1-\alpha} K_{gt}^\theta, \quad (1)$$

where  $0 < \alpha < 1$  and  $0 < \theta < 1 - \alpha$ . The condition  $\theta < 1 - \alpha$  prohibits the possibility of endogenous growth (see Barro and Sala-i-Martin [1995], 153). In equation (1),  $K_t$  and  $K_{gt}$  denote private and public capital at time  $t$ , respectively, and  $z_t l_t$  represents the quantity of skill-weighted labor input. We allow for exogenous technical progress in the use of labor input,  $l_t$ , so that  $z_t$  grows at a constant rate over time:

$$z_t = \gamma_z z_{t-1}, \gamma_z > 1, \text{ and } z_0 = 1. \quad (2)$$

This feature of the technology will make it possible for the economy to experience balanced growth in the long run at a rate related to  $\gamma_z$ . We treat  $K_{gt}$  as a pure public good that enhances each firm's production.<sup>4</sup>

We assume that public investments are financed by a flat tax on income,  $0 < \tau_t < 1$ , that can vary with time. Hence, we can express new outlays of public capital,  $K_{gt+1}$ , as

$$K_{gt+1} = \underbrace{\tau_t Y_t}_{\text{Public Investment}} + (1 - \delta) K_{gt}, \quad (3)$$

$$K_{g0} > 0 \text{ given,}$$

where  $0 < \delta < 1$ . Observe that  $\tau_t$  also represents the share of gross public investment in GDP. Our main objective is to characterize the efficient ratio of public investment to output.

### Firms

Let  $r_t$  and  $W_t$  denote, respectively, the rental price of private capital and the wage at date  $t$ . Then, taking as given the sequence  $\{r_t, W_t\}_{t=0}^\infty$ , each firm maximizes profits by solving

$$\max_{K_t, l_t} K_t^\alpha (z_t l_t)^{1-\alpha} K_{gt}^\theta - r_t K_t - W_t l_t. \quad (4)$$

The first order conditions associated with this problem imply that

$$r_t = \alpha K_t^{\alpha-1} (z_t l_t)^{1-\alpha} K_{gt}^\theta \quad (5)$$

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<sup>4</sup> Thus, we abstract from congestion considerations and think of the aggregate stock of public capital as being available to all firms. This assumption typically subjects the economic environment to scale effects with respect to steady state allocations since public infrastructure is non-rival (see Barro and Sala-i-Martin [1992]).

and

$$W_t = (1 - \alpha)K_t^\alpha z_t^{1-\alpha} l_t^{-\alpha} K_{gt}^\theta. \quad (6)$$

### Households

The economy is inhabited by a large number of identical households. Their preferences are given by

$$\mathcal{U} = \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}, \text{ with } \sigma > 0, \quad (7)$$

where  $0 < \beta < 1$  is a subjective discount rate. At each date, households decide how much to consume or save and how much capital to rent to firms. Each household is assumed to be endowed with one unit of time that they supply inelastically. Their budget constraint is given by

$$C_t + I_t = (1 - \tau_t) [W_t l_t + r_t K_t], \quad (8)$$

where

$$\begin{aligned} K_{t+1} &= I_t + (1 - \delta)K_t, \\ K_0 &> 0 \text{ given.} \end{aligned} \quad (9)$$

In equation (8),  $C_t$  and  $I_t$  denote consumption and gross investment, respectively. Before proceeding with the household's problem, we find it useful to first derive the economy's constant balanced growth rate in the steady state. This will allow us to define a normalization of the economy's variables that will show explicitly why the optimal rate of public investment may depend on the exogenous rate of technical progress,  $\gamma_z > 1$ . We denote the long-run growth rate of a given variable  $x$  by  $\gamma_x$ .

Equation (3) implies that if public investment ultimately grows at the constant rate  $\gamma_{I_g}$ , then  $\gamma_{I_g} = \gamma_{K_g}$ . In addition, the household's budget constraint (8) implies that  $\gamma_C = \gamma_I = \gamma_{I_g} = \gamma_Y$  so that  $\gamma_{K_g} = \gamma_Y$ .<sup>5</sup> Since equation (9) also means that  $\gamma_I = \gamma_K$ , we further have that  $\gamma_K = \gamma_Y$ . Therefore, in the end, all variables in the economy grow at the common growth rate  $\gamma_Y$ . To determine this growth rate, observe from the production technology described in (1) that

$$\frac{Y_{t+1}}{Y_t} = \left( \frac{K_{t+1}}{K_t} \right)^\alpha \gamma_z^{1-\alpha} \left( \frac{K_{gt+1}}{K_{gt}} \right)^\theta$$

or

$$\gamma_Y = \gamma_K^\alpha \gamma_z^{1-\alpha} \gamma_{K_g}^\theta.$$

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<sup>5</sup> Note that  $W_t l_t + r_t K_t = Y_t$ .

Since  $\gamma_{K_g} = \gamma_Y = \gamma_K$  above, we can immediately obtain the economy's balanced growth rate, which we simply denote by  $\gamma$ :

$$\gamma = \gamma_z^{\frac{1-\alpha}{1-\alpha-\theta}}. \quad (10)$$

Given this growth rate, a sensible normalization for our economy is one that expresses our model's variables in detrended form. Specifically, for any variable  $X_t$  that grows at rate  $\gamma$  along the balanced growth path, we express its detrended counterpart in lowercase form:

$$x_t = \frac{X_t}{\gamma^t}.$$

In the steady state, detrended variables will then be constant, while non-detrended variables will grow at the constant rate,  $\gamma$ .<sup>6</sup>

Taking the sequence of prices  $\{r_t, w_t\}_{t=0}^\infty$ , as well as the sequence of tax rates  $\{\tau_t\}_{t=0}^\infty$ , as given, the household's problem may now be expressed as<sup>7</sup>

$$\max_{\{c_t, k_{t+1}\}_{t=0}^\infty} \mathcal{U} = \sum_{t=0}^{\infty} \underbrace{(\beta \gamma^{(1-\sigma)})^t}_{\beta^*} \frac{c_t^{1-\sigma}}{1-\sigma}, \quad (11)$$

subject to

$$\begin{aligned} c_t + \gamma k_{t+1} - (1 - \delta)k_t &= (1 - \tau_t)[w_t l_t + r_t k_t], \\ k_0 &> 0 \text{ given.} \end{aligned} \quad (12)$$

The solution to the household's dynamic optimization problem yields the familiar Euler equation,

$$\gamma c_t^{-\sigma} = \beta^* c_{t+1}^{-\sigma} [(1 - \tau_{t+1})r_{t+1} + 1 - \delta]. \quad (13)$$

In the above expression, taxes distort private incentives to consume and save but also finance public infrastructure that raises future returns to private investment; recall that  $r_{t+1} = \alpha k_{t+1}^{\alpha-1} k_{g,t+1}^\theta$ . Given this trade-off, how should society choose the share of public investment in output? An intuitive answer to this question might be to select the ratio of public investment to output that maximizes steady state welfare. This captures perhaps the simplest notion of optimal policy. We show that such intuition indeed replicates conventional wisdom obtained from endogenous growth frameworks. However, in both our model and endogenous growth settings, ignoring transition dynamics turns out to have crucial implications for policy.

<sup>6</sup> Observe that if  $X_t$  grows at the rate  $\gamma$ , then  $X_t = \gamma^t X_0$ . Therefore  $x_t = \frac{X_t}{\gamma^t} = X_0$ .

<sup>7</sup> It should be noted that the modified discount rate,  $\beta^*$ , implicitly imposes a restriction on the extent of technical progress, since  $\gamma^{(1-\sigma)}$  must be less than  $1/\beta$  in order that the maximization problem be well defined.

### The Decentralized Steady State Equilibrium and a Policy Golden Rule

We can describe the decentralized steady state equilibrium of the environment laid out in this section most easily as a vector  $\{c, i, k, k_g\}$  such that, given a constant share of public investment in output,  $\tau$ ,<sup>8</sup>

$$A) \quad \gamma = \beta^* [(1 - \tau)\alpha k^{\alpha-1} k_g^\theta + 1 - \delta],$$

$$B) \quad c + i = (1 - \tau)k^\alpha k_g^\theta,$$

$$C) \quad [\gamma - (1 - \delta)]k = i,$$

$$D) \quad [\gamma - (1 - \delta)]k_g = \tau k^\alpha k_g^\theta.$$

Conditions A) and B) are the households' long-run consumption/savings decision and budget constraint, respectively. Conditions C) and D) are the normalized accumulation equations for private and public capital.

By combining conditions A) and D), and given the definition of  $\beta^*$ , it is straightforward to show that the decentralized equilibrium ratio of public to private capital satisfies

$$\frac{k_g}{k} = \frac{\left[\frac{\gamma^\sigma}{\beta} - (1 - \delta)\right]\tau}{\alpha [\gamma - (1 - \delta)](1 - \tau)}. \quad (14)$$

In particular, the higher the tax rate, the more public investment takes place and the higher the steady state ratio of public to private capital. Furthermore, one can show that the detrended level of private capital in the steady state is given by

$$k^{1-\alpha-\theta} = \left[ \frac{(1 - \tau)\alpha}{\frac{\gamma^\sigma}{\beta} - (1 - \delta)} \right]^{1-\theta} \left[ \frac{\tau}{\gamma - (1 - \delta)} \right]^\theta. \quad (15)$$

Observe in the equation above that  $k$  is a strictly concave function of  $\tau$ . In particular, setting  $\tau = \theta$  maximizes the steady state quantity of private capital. Furthermore, we can make use of conditions A), B), and C) to solve for consumption. Given the preferences in (7), it follows that steady state welfare is

$$\mathcal{U} = \frac{\left\{ \frac{1}{\alpha} \left[ \frac{\gamma^\sigma}{\beta} - (1 - \delta) \right] - [\gamma - (1 - \delta)] \right\} k^{(1-\sigma)}}{(1 - \beta\gamma^{(1-\sigma)})(1 - \sigma)}, \quad (16)$$

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<sup>8</sup> We use the expression "decentralized" to emphasize that steady state allocations obtain from household and firm optimization conditional on policy.

so that  $\mathcal{U}$  depends on  $\tau$  only through private capital. Because  $\mathcal{U}$  is a strictly increasing function of  $k$ , while  $k$  is concave in  $\tau$ , setting  $\tau = \theta$  also maximizes steady state welfare.

To recapitulate, we have just shown in an exogenous growth context that along the balanced growth path, the optimal share of gross public investment in output,  $\tau$ , must equal the public capital elasticity of output,  $\theta$ . As it happens, this is equivalent to the *full* optimizing solution in an endogenous growth model without transition dynamics. Since endogenous growth requires  $\alpha + \theta = 1$ , and since  $\alpha$  is approximately 0.36 in the U.S. economy, endogenous growth models imply that the optimal share of public investment in output should be 0.64, an implausibly large number.<sup>9</sup> Nothing in our exogenous growth framework requires that  $\alpha + \theta = 1$ , and consequently,  $\tau$  does not have to be implausibly large at  $1 - \alpha$ .

There remains, however, another empirical puzzle to solve. Most estimates would place  $\theta$  between 0.05 and 0.15. In contrast, we have seen that the share of gross public investment in output ( $\tau$  in our model) has consistently remained only around 0.03 over the postwar period. To address this puzzle, we now study the full welfare-maximizing policy problem that takes into account both steady state considerations and transition dynamics.

### **3. EFFICIENT PUBLIC INVESTMENT WITH COMMITMENT: THE RAMSEY PROBLEM**

Unfortunately, there is no simple way to characterize the Ramsey problem for our economy. Setting up and solving the full welfare maximization problem, however, allows us to address explicitly two important notions associated with efficient policy.

First, we show why the efficient sequence of public investment is not time consistent. Second, we explain why Woodford's (1999) "timeless perspective" provides potentially one answer to this problem. In essence, the timeless perspective advocates the implementation of the steady state solution to the Ramsey problem at all dates. It thus considers optimality from the vantage point of a date far in the past. Because the implied policy is one to which any welfare-maximizing government would have wished to commit itself on that date in the past, this solution concept avoids the problem of time consistency. We will now address these issues in detail.

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<sup>9</sup> An implicit assumption here is that the share of private capital is measured adequately. An alternative model that allowed for investment in human capital might allow  $\theta$  to be significantly less than 0.64. Indeed, with a broader concept of capital that allowed for not only physical but also human capital, the share of capital in output might be closer to 0.75 (see Barro and Sala-i-Martin [1992, 38]). Observe also that in an endogenous growth model that allowed for transition dynamics (i.e., where the balanced growth path was reached asymptotically), the full welfare-optimizing solution would not necessarily prescribe  $\tau = \theta$  at all dates.

Consider a benevolent government that, at date zero, is concerned with choosing a sequence of tax rates consistent with the development of public infrastructure that maximizes household welfare. In choosing policy, this government takes as given the decentralized behavior of firms and households in the spirit of Chamley (1986). We further assume that at date zero, it can credibly commit to any sequence of policy actions. The problem faced by this benevolent government would then be to maximize (11) subject to equations (12), (13), and (3) (in normalized form), and the corresponding Lagrangean can be written as

$$\begin{aligned} \max_{\{c_t, \tau_t, k_{t+1}, k_{gt+1}\}_{t=0}^{\infty}} \mathcal{L} = & \sum_{t=0}^{\infty} \beta^{*t} \frac{c_t^{1-\sigma}}{1-\sigma} \\ & + \sum_{t=0}^{\infty} \beta^{*t} \mu_{1t} \left\{ \beta^* c_{t+1}^{-\sigma} [(1 - \tau_{t+1}) r_{t+1} + 1 - \delta] - \gamma c_t^{-\sigma} \right\} \\ & + \sum_{t=0}^{\infty} \beta^{*t} \mu_{2t} [\tau_t y_t + (1 - \delta) k_{gt} - \gamma k_{gt+1}] \\ & + \sum_{t=0}^{\infty} \beta^{*t} \mu_{3t} [(1 - \tau_t) y_t + (1 - \delta) k_t - \gamma k_{t+1} - c_t], \end{aligned} \quad (17)$$

where  $y_t = k_t^\alpha l_t^{1-\alpha} k_{gt}^\theta$  and  $r_t = \alpha(y_t/k_t)$ .

The first constraint in (17) makes clear that our benevolent planner takes households' consumption/savings behavior as given. It can, however, influence the intertemporal allocations they choose by altering tax policy over time. The optimal selection of  $\tau_t$  is governed by the following two equations:

$$\frac{\partial \mathcal{L}}{\partial \tau_t} : \mu_{20} - \mu_{30} = 0 \text{ for } t = 0 \quad (18)$$

and

$$\frac{\partial \mathcal{L}}{\partial \tau_t} : \mu_{2t} - \mu_{3t} - \mu_{1t-1} \frac{\alpha c_t^{-\sigma}}{k_t} = 0 \quad \forall t > 0. \quad (19)$$

The fact that these first order conditions differ for  $t = 0$  and  $t > 0$  suggests an incentive to take advantage of initial conditions in the first period with the promise never to do so in the future. It is exactly in this sense that the optimal policy is not time consistent, since once date zero has passed, a new planner wishing to solve for the optimal policy at some date  $t > 0$  would always choose a tax rate on that date different from what had been prescribed at time zero. For the purpose of this section, therefore, we imagine that the optimization takes place only once, in period zero. Once our benevolent planner has decided on a course of action, his hands are tied and he is precommitted to that course of action.

It should be noted that the choice of  $\tau_t$  introduces a lagged predetermined variable,  $\mu_{1,t-1} \geq 0$ . At a purely mechanical level, the corresponding initial condition,  $\mu_{1,-1}$ , serves as an artificial device that helps make stationary the final system of difference equations that characterizes the optimal solution. However, unlike with fundamental state variables such as the private or public capital stock, this initial condition is not arbitrary. Instead, since the optimal choice of  $\tau_0$  should satisfy equation (18), it must be the case that  $\mu_{1,t-1} = 0$  in (19) at  $t = 0$ . Alternatively, Dennis (2001, 6) points out that the lagged Lagrange multiplier,  $\mu_{1,t-1}$ , may be interpreted as the “current value of promises not to exploit the initial state” and, in particular, to abide by past commitments. However, since no history exists prior to period zero, there are no past commitments on which to assign any value at that date. It is optimal, therefore, to set  $\mu_{1,-1} = 0$ .

The fact that the optimal policy is chosen once and for all in period zero does not necessarily imply that it is not flexible. On the contrary, the solution to the Ramsey problem provides a description of where to set  $\tau_t$  in every state of the world. We shall see that it is also explicit about how the share of public investment in output depends on the economic environment. Therefore, as noted in Dennis (2001, 6), “if a change to one or more parameters takes place, the policy rule automatically reflects this change; [and] there is no need for re-optimization to take place.” Accordingly, the remaining first order conditions associated with problem (17) are

$$\frac{\partial \mathcal{L}}{\partial c_t} : c_0^{-\sigma} + \sigma \gamma \mu_{10} c_0^{-\sigma-1} - \mu_{30} = 0 \text{ for } t = 0,$$

and,  $\forall t > 0$ ,

$$\frac{\partial \mathcal{L}}{\partial c_t} : c_t^{-\sigma} - \sigma \mu_{1t-1} c_t^{-\sigma-1} [r_t(1 - \tau_t) + 1 - \delta] + \sigma \gamma \mu_{1t} c_t^{-\sigma-1} - \mu_{3t} = 0, \quad (20)$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial k_{t+1}} : & \mu_{1t} \beta^* c_{t+1}^{-\sigma} \alpha (\alpha - 1) \left( \frac{y_{t+1}}{k_{t+1}^2} \right) (1 - \tau_{t+1}) + \beta^* \mu_{2t+1} \tau_{t+1} r_{t+1} \\ & - \mu_{3t} \gamma + \beta^* \mu_{3t+1} [(1 - \tau_{t+1}) r_{t+1} + 1 - \delta] \\ & = 0, \end{aligned} \quad (21)$$

and

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial k_{gt+1}} : & \mu_{1t} \beta^* c_{t+1}^{-\sigma} \alpha \theta \left( \frac{y_{t+1}}{k_{t+1} k_{gt+1}} \right) (1 - \tau_{t+1}) - \mu_{2t} \gamma \\ & + \beta^* \mu_{2t+1} \left[ \tau_{t+1} \theta \frac{y_{t+1}}{k_{gt+1}} + 1 - \delta \right] + \beta^* \mu_{3t+1} (1 - \tau_{t+1}) \theta \left( \frac{y_{t+1}}{k_{gt+1}} \right) \\ & = 0. \end{aligned} \quad (22)$$

With these first order conditions in hand, we now turn to the long-run properties of the efficient solution for public investment.

### The Timeless Perspective and a Modified Policy Golden Rule

As we have already seen, one of the features associated with the optimal policy described in this section is that it requires the policymaker to commit once and for all to his chosen rule at time zero. That is, optimal commitments are generally not time consistent. However, in the context of monetary policy, Woodford (1999) points out that “the optimal commitment fails to be time consistent only if the central bank considers ‘optimality’ at each point in time in a way that allows it to consider the advantages, from the vantage point of that particular moment, of a policy change at that time that was not previously anticipated.” One way around this problem, therefore, would be for the policymaker to adopt a pattern of behavior “to which it would have wished to commit itself to at a date far in the past, contingent upon the random events that have occurred in the meantime.” Woodford refers to this pattern of behavior as the “timeless perspective.”

The attraction of Woodford’s timeless perspective is that it retains the solution to the Ramsey problem as the concept of optimal policy while getting rid of the unique nature of date zero. Two potential problems with the notion of timeless perspective are that it may be consistent with multiple policy outcomes (see Dennis 2001) and that welfare may be lower than that obtained under alternative time consistent solutions (see Jensen and McCallum 2002).

Under the timeless perspective, the policy rule must ensure that the optimal stationary equilibrium is eventually reached or, if already reached, that it continues in that state. In our context, the optimal stationary equilibrium is given by a vector  $\{c, y, \tau, k, k_g, \mu_1, \mu_2, \mu_3\}$  that solves equations (19) through (22), along with the Euler equation (13), the resource constraint (12), the equation describing the evolution of public capital (3) (in normalized form), and the definition of output,  $y_t = k_t^\alpha l_t^{1-\alpha} k_{gt}^\theta$ , all without time subscripts. The optimal steady state equilibrium, therefore, can be characterized as a system of eight equations in eight unknowns, which is shown in detail in the Appendix.

In the steady state, households’ optimal consumption/savings decisions satisfy the familiar Euler equation,

$$\gamma - \beta^* \left[ (1 - \tau)\alpha \left( \frac{y}{k} \right) + 1 - \delta \right] = 0. \quad (23)$$

Furthermore, the Appendix shows that the efficient share of public investment in output must be such that

$$\gamma - \beta^* \left[ \theta \left( \frac{y}{k_g} \right) + 1 - \delta \right] = 0. \quad (24)$$

It follows from equations (23) and (24) that the Ramsey solution equates the after-tax return to private investment,  $(1 - \tau)\alpha(y/k)$ , with the marginal return to public investment,  $\theta(y/k_g)$ . Consequently, we can immediately pin down the ratio of public to private capital, given  $\tau$ , as

$$\frac{k_g}{k} = \frac{\theta}{\alpha(1 - \tau)}. \quad (25)$$

As expected, the higher the tax rate, the more public capital can be generated relative to private capital.

The idea that the after-tax return to private investment must equal the marginal return to public investment at the optimum also helps determine the efficient share of public investment in output. Note that in our framework the opportunity cost of one unit of resources invested in the public sector is the after-tax return this unit would have otherwise earned in the private sector, or, by equation (23),

$$\underbrace{(1 - \tau)\alpha\left(\frac{y}{k}\right)}_{\text{return to private investment}} = \frac{\gamma}{\beta^*} - (1 - \delta). \quad (26)$$

The marginal benefit of investing one unit of resources in the public sector is  $\theta(y/k_g)$ , and since public capital accumulates according to the law of motion  $k_g[\gamma - (1 - \delta)] = \tau y$  in the steady state, we have

$$\theta\left(\frac{y}{k_g}\right) = \theta\left(\frac{\gamma - (1 - \delta)}{\tau}\right). \quad (27)$$

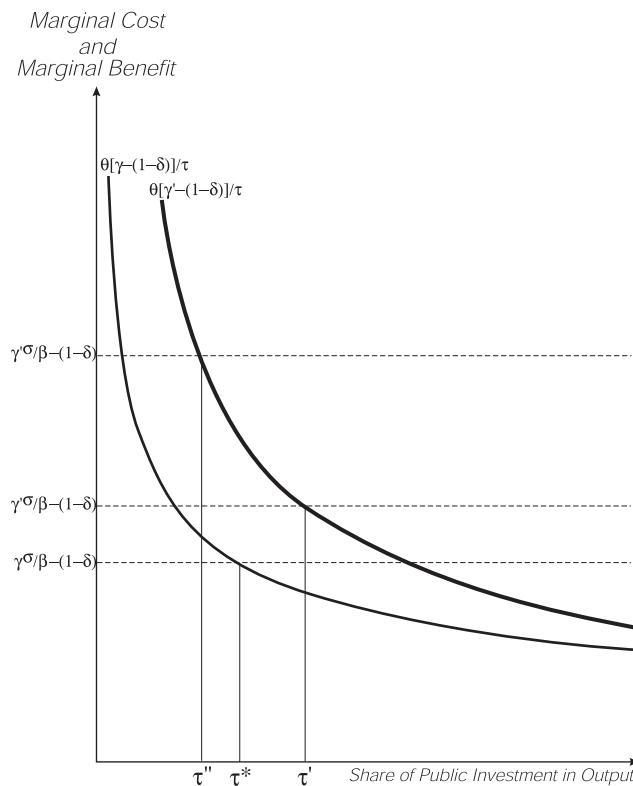
Hence, equating marginal cost and marginal benefit (i.e., the RHS of equations 26 and 27) directly yields the efficient solution for public investment as a fraction of output:

$$\tau = \theta \left\{ \frac{\gamma - (1 - \delta)}{\frac{\gamma^\sigma}{\beta} - (1 - \delta)} \right\} < \theta. \quad (28)$$

We can think of the equation above as a modified golden rule for policy. Analogous to the modified golden rule for private capital in the one sector growth model, the share of public investment in output given by (28) falls short of the policy golden rule outlined in the previous section by an amount that depends importantly on discounting. To see this, observe that when  $\delta = 1$  in equation (28),  $\tau = (\beta\gamma^{1-\sigma})\theta = \beta^*\theta < \theta$ . Moreover, from (14) and (25), it would then be the case that

$$\underbrace{\frac{k_g}{k} = \frac{\theta}{\alpha(1 - \beta^*\theta)}}_{\text{Modified Policy Golden Rule}} < \underbrace{\frac{\theta}{\alpha\beta^*(1 - \theta)}}_{\text{Policy Golden Rule}} = \frac{k_g}{k}. \quad (29)$$

In other words, although the policy golden rule eventually leads to more public infrastructure, and although public capital matters in production, the impatience reflected in the rate of time preference means that it is not optimal to

**Figure 2 Efficient Public Investment**

reduce current consumption through higher taxes to reach this higher ratio of public to private capital.

Relative to most earlier work, the modified policy golden rule condition is an important one in at least two respects. First, it implies that a high elasticity of output with respect to public capital,  $\theta$ , does not necessarily have to translate into a large share of public investment in output,  $\tau$ . Therefore, observed empirical estimates of  $\theta$  lying between 0.05 and 0.15 do not have to be inconsistent with the 0.03 share of public investment in GDP. Second, the efficient share of public investment in GDP now depends on a variety of preference and technology considerations, including exogenous productivity growth, the rate of depreciation, and the coefficient of intertemporal substitution.

### Comparative Statics

Figure 2 depicts the effects of a rise in exogenous labor productivity growth on the efficient ratio of public investment to output. We can see that an increase in the rate of labor augmenting technical progress from  $\gamma$  to  $\gamma'$  raises both the marginal cost,  $\frac{\gamma}{\beta^*} - (1 - \delta)$ , and the marginal benefit,  $\theta \left( \frac{\gamma - (1 - \delta)}{\tau} \right)$ , of public investment. Intuitively, the returns to both types of capital increase, and it is not clear whether investment in public capital should increase or fall. Depending on the degree to which the marginal cost of public investment rises in Figure 2, we can see that the optimal rate of public investment,  $\tau^*$ , may rise to  $\tau'$  or instead decrease to  $\tau''$ . In the case with 100 percent depreciation, equation (28) reduces to  $\tau^* = \beta \gamma^{(1-\sigma)} \theta$ . Hence, it follows that

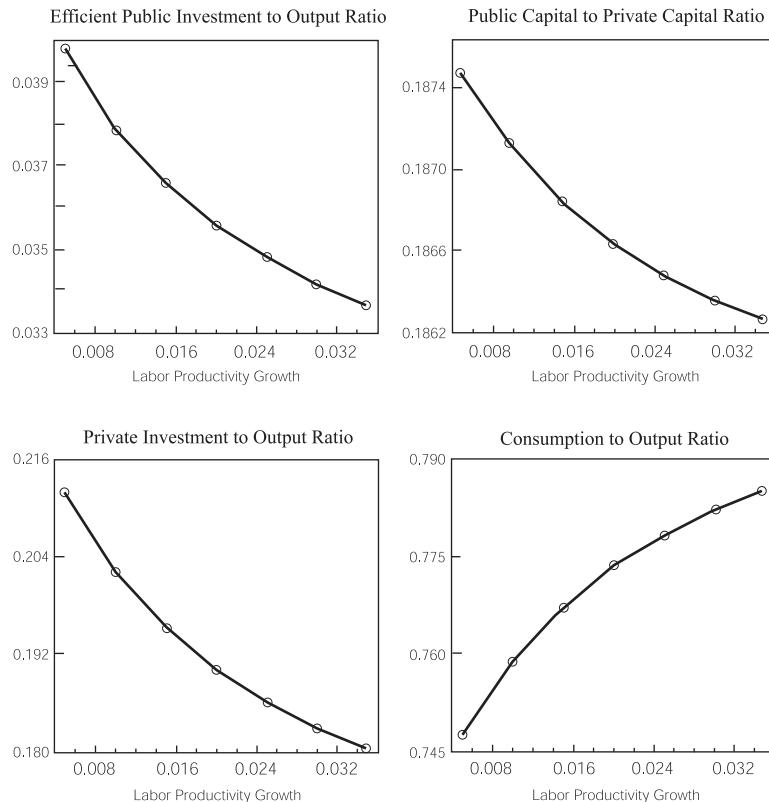
$$\frac{\partial \tau^*}{\partial \gamma} = (1 - \sigma) \beta^{-\sigma} \theta \geqslant 0 \Leftrightarrow \frac{1}{\sigma} \geqslant 1. \quad (30)$$

Put another way, when households are not particularly willing to substitute consumption across time (i.e.,  $1/\sigma < 1$ ), an increase in exogenous labor productivity growth leads them to want to increase present consumption relative to output. Therefore, with fewer resources available for investment, the optimal steady state share of public investment in output must fall.

To conclude this section, we calibrate the above example in order to determine what value of  $\theta$  is implied by the theory. We shall think of our benchmark as that of an economy resembling the United States. Other than for the public capital elasticity of output,  $\theta$ , the parameter values of the economy are selected along the lines of conventional general equilibrium quantitative studies. Thus, a time period represents one quarter, and we set the subjective discount rate,  $\beta$ , to 0.99. The share of private capital in output,  $\alpha$ , is set to 1/3, and  $\sigma$  is chosen so as to make the coefficient of intertemporal substitution 1/2. Capital is assumed to depreciate at a 10 percent annual rate. We let the rate of technical progress vary so that the rate of growth in per capita output lies between zero and 3.5 percent. In the United States, the rate of growth of per capita GDP has averaged 2.5 percent since World War II.

Given these parameter values, equations (28) and (29) indicate that the choice of  $\theta$  simultaneously pins down two measures, namely, the share of public investment in GDP and the ratio of public to private capital. There is an obvious sense, then, in which our model can fail, since choosing  $\theta$  to match one measure for the United States would leave it free to miss its mark on the other. We find that setting the public capital elasticity of output to 0.06 helps generate a share of public investment in output between 3.3 and 4 percent as in U.S. data. This is shown in the upper left-hand panel of Figure 3. Interestingly, we find that this same public capital elasticity of output leads to a ratio of public to private capital that matches well U.S. postwar experience at approximately 20 percent (see the upper right-hand panel of Figure 3). We conclude that if the share of public investment and the ratio of public to

**Figure 3 Steady State Responses to Changes in Labor Productivity Growth**



private capital are approximately efficient in the United States, then the public capital elasticity of output implied by the theory lies in the lower range of most empirical estimates at around 0.06.

Figure 3 also shows that the efficient share of public investment in GDP falls with increases in exogenous labor productivity growth. At the same time, the consumption to output ratio rises in the lower right-hand panel of Figure 3. In this calibrated example, as exogenous labor productivity growth increases, households can afford to consume more relative to output and both private and public investment as a fraction of GDP decreases.

#### 4. SUMMARY

In a setting where public infrastructure plays a productive role, we have characterized the main features of efficient public investment under commitment. In contrast to most previous studies, we have shown that the optimal share of public investment in output may be less than its elasticity in the long run and, moreover, that this share may depend in important ways on assumed preferences and technology. We have also stressed the crucial nature of the commitment assumption, as the optimal sequence of public investments is not time consistent. Finally, if the observed ratios of public investment to GDP and of public to private capital are approximately optimal in the United States, then our model suggests that the government capital elasticity of output is likely to be relatively low at 0.06.

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#### APPENDIX

As described in the text, the optimal stationary equilibrium in our framework is a vector  $\{c, y, \tau, k, k_g, \mu_1, \mu_2, \mu_3\}$  that solves eight equations. The first four equations are

$$y - k^\alpha l^{1-\alpha} k_g^\theta = 0 \quad (31)$$

$$\gamma - \beta^* \left[ (1 - \tau) \alpha \left( \frac{y}{k} \right) + 1 - \delta \right] = 0 \quad (32)$$

$$\tau y + (1 - \delta) k_g - \gamma k_g = 0 \quad (33)$$

$$(1 - \tau)y + (1 - \delta)k - \gamma k - c = 0. \quad (34)$$

The next four equations are worked out in more detail.

Equation (21) reads as:

$$\begin{aligned} & \mu_1 \beta^* c^{-\sigma} \alpha (\alpha - 1) \left( \frac{y}{k^2} \right) (1 - \tau) + \beta^* \mu_2 \tau \alpha \left( \frac{y}{k} \right) \\ & - \mu_3 \left[ \underbrace{\gamma - \beta^* \left[ (1 - \tau) \alpha \left( \frac{y}{k} \right) + 1 - \delta \right]}_{= 0 \text{ by equation (32)}} \right] \\ & = 0. \end{aligned}$$

Thus, we have

$$\mu_1 \beta^* c^{-\sigma} \alpha (\alpha - 1) \left( \frac{y}{k^2} \right) (1 - \tau) + \beta^* \mu_2 \tau \alpha \left( \frac{y}{k} \right) = 0,$$

or more simply

$$\left(\frac{\mu_1 c^{-\sigma}}{k}\right)(\alpha - 1)(1 - \tau) + \mu_2 \tau = 0. \quad (35)$$

In the steady state, equation (19) reads as

$$-\mu_1 c^{-\sigma} \alpha \left(\frac{y}{k}\right) + \mu_2 y - \mu_3 y = 0,$$

so that

$$\left(\frac{\mu_1 c^{-\sigma}}{k}\right) \alpha = \mu_2 - \mu_3. \quad (36)$$

Equation (20) is

$$c^{-\sigma} - \sigma \mu_1 c^{-\sigma-1} \left[ (1 - \tau) \alpha \left(\frac{y}{k}\right) + 1 - \delta \right] + \sigma \gamma \mu_1 c^{-\sigma-1} - \mu_3 = 0,$$

or

$$c^{-\sigma} - \sigma \mu_1 c^{-\sigma-1} \left\{ \left[ (1 - \tau) \alpha \left(\frac{y}{k}\right) + 1 - \delta \right] - \gamma \right\} - \mu_3 = 0,$$

so that by equation (32),

$$c^{-\sigma} - \sigma \mu_1 c^{-\sigma-1} \gamma \left[ \frac{1}{\beta^*} - 1 \right] - \mu_3 = 0. \quad (37)$$

Finally, equation (22) reads as

$$\begin{aligned} \mu_1 \beta^* c^{-\sigma} \left[ \frac{\alpha \theta y}{kk_g} \right] (1 - \tau) - \mu_2 \gamma + \beta^* \mu_2 \left[ \tau \theta \left(\frac{y}{k_g}\right) + 1 - \delta \right] \\ + \beta^* \mu_3 (1 - \tau) \theta \left(\frac{y}{k_g}\right) = 0. \end{aligned}$$

This simplifies to

$$\begin{aligned} \beta^* \theta \left(\frac{y}{k_g}\right) (1 - \tau) (\mu_2 - \mu_3) - \mu_2 \gamma + \beta^* \mu_2 \left[ \tau \theta \left(\frac{y}{k_g}\right) + 1 - \delta \right] \\ + \beta^* \mu_3 (1 - \tau) \theta \left(\frac{y}{k_g}\right) = 0 \end{aligned}$$

by equation (36), which implies

$$\beta^* \theta \left(\frac{y}{k_g}\right) (1 - \tau) - \gamma + \beta^* \left[ \tau \theta \left(\frac{y}{k_g}\right) + 1 - \delta \right] = 0,$$

or

$$\gamma - \beta^* \left[ \theta \left(\frac{y}{k_g}\right) + 1 - \delta \right] = 0. \quad (38)$$

This last equation and equation (32) imply that the Ramsey solution equates the marginal return to government capital with the after-tax return to private capital.

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