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# MONEY AND THE GAIN FROM ENDURING RELATIONSHIPS IN THE TURNPIKE MODEL

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**Abstract**

This paper presents a stochastic version of Townsend's turnpike model in which the aggregate endowment is distributed randomly between two sets of agents and in which agents of each type are allowed to remain at a trading post for multiple periods. Agents use money as a means of exchange when they meet as strangers but use private securities when they remain paired at the same trading post. Both welfare and the income velocity of money increase monotonically with the length of the trading session.

## I. Introduction

With his turnpike model of exchange, Townsend (1980) demonstrates that noninterest-bearing fiat money is valued in equilibrium when agents' itineraries between spatially separated trading posts completely rule out the existence of markets for private debt. In contrast to Townsend's model are those featuring a full set of Arrow-Debreu securities, where trade is sufficiently centralized so as to never require the use of a low-yielding outside asset as a means of payment. This paper presents a model intermediate to these two extremes, in which some trades are effected through the transfer of private securities while others are possible only when government-issued money is available. Specifically, it modifies the turnpike environment by randomly distributing the aggregate endowment between two sets of agents and by allowing agents of each type to remain at a trading post for multiple periods.

Comparing Townsend's original turnpike model to the Arrow-Debreu framework suggests a distinction between money, which is used to facilitate trade between agents who meet as strangers in isolated markets, and private securities, which are traded among agents who meet repeatedly in a centralized market. The modified turnpike model illuminates this distinction. Agents in the modified turnpike model use money when they meet as strangers but use private securities when they remain paired at the same trading post. The importance of money in the modified turnpike economy, therefore, depends directly on the frequency with which agents meet as strangers and inversely on the length of each multi-period trading session.

In fact, the length N of the trading session indicates exactly where each turnpike model lies between the two extremes. Those economies with N close to unity resemble Townsend's original turnpike economy: agents are frequently meeting as strangers and money plays an important role in facilitating trade. Those economies with larger N become increasingly similar to the Arrow-Debreu case: trade becomes more centralized, the scope for private debt expands, and the role of money diminishes.

There are two other respects in which the modified turnpike economies are intermediate to Townsend's pure monetary model on the one hand and the Arrow-Debreu model on the other, with the session length N measuring distances between the two extremes. First, Gale (1978) notes that the differences between monetary and Arrow-Debreu models are summarized by the form of an agent's budget constraints. In a pure monetary environment like Townsend's, each agent faces a sequence of budget constraints that must be balanced period-by-period. In an Arrow-Debreu setting, each agent faces a single lifetime budget constraint. Agents in the modified turnpike model face a sequence of budget constraints, but each constraint requires only that expenditures and receipts balance over the entire course of a multi-period trading session. Thus, as the trading session is lengthened, the constraints look less like those in a pure monetary environment and more like those in an Arrow-Debreu environment.

Second, Hahn (1973) shows that competitive equilibria in pure monetary economies are generally not Pareto optimal, whereas equilibria in Arrow-Debreu economies typically are. In the modified turnpike model, welfare increases monotonically and approaches the level achieved by Pareto optimal allocations as N grows larger. Again, the modified turnpike model is seen to be intermediate to the two extremes.

While the implications of longer trading periods in the turnpike model are explored in great detail by Manuelli and Sargent (1992), the periodic, deterministic endowment specification that they employ gives rise to peculiar nonmonotonicities as the trading session is lengthened. For example, when the length of the trading period coincides with the length of the cycle in the endowment process, the gains from trade can be completely exhausted without government-issued money. But when the trading session is extended one period beyond the length of the endowment cycle, money is again needed. In this case, an increase in session length yields an increase in the demand for money, obscuring the connection between the value of money and the prevalence of trade among strangers. The additional stochastic endowment feature introduced here restores this connection, so that the turnpike economies clearly represent cases intermediate to pure monetary and Arrow-Debreu environments, with the session length N measuring distances between the two extremes.

The modified turnpike model is specified in the next section. Section III describes both nonmonetary and monetary competitive equilibria in the modified turnpike environment; agents' consumption patterns in these equilibria are compared to the consumption patterns provided by Pareto optimal resource allocations. Section IV explores how competitive equilibria change as the trading session is lengthened. Section V concludes.

## II. The Modified Turnpike Model

Following Townsend (1980) and Manuelli and Sargent (1992), the economy is assumed to consist of a large number of infinitely-lived agents who are distributed uniformly into a countable number of trading posts located at discrete intervals along a turnpike of infinite length. These trading posts are spatially isolated, meaning that agents at one post can neither trade nor communicate with agents at other posts.

The agents are of two types, labelled i=A and i=B. At any given time t=1,2,3..., there are equal numbers of each type at each trading post. At the end of each period t=N,2N,3N,..., where $N \geq 1$, each agent of type A leaves his trading post, arriving at the next post to the east along the turnpike at the beginning of the following period. Simultaneously, each type B agent moves one market to the west. The length N of a trading session, therefore, measures the frequency with which agents meet as strangers along the turnpike.

In each period t=1,2,3,..., an endowment shock $s_t \in \{a,b\}$ is revealed to the agents. The timing of this revelation is such that when an agent is moving between two posts he leaves the first before, but arrives at the second after, $s_t$ becomes known. If $s_t=a$, then each type A agent in the economy receives an endowment of one unit of a perishable consumption good in period t, while type B agents receive no endowment. If $s_t=b$, then each type B agent receives a unit of the good in period t, while type A agents get nothing. The shock $s_t$ is iid over time and has a binomial distribution with equal probability of either outcome.

Preferences over state-contingent consumption plans are common to all agents and are represented by the additively time-separable expected utility function

$$E\left\{\sum_{t=1}^{\infty}\beta^{t}u(c_{t})\right\},$$

where $\beta\in(0,1)$ is the discount factor, $c_t$ is consumption at date t, and $u(\cdot)$ is strictly increasing, strictly concave, and twice continuously differentiable with $\lim_{c\to 0} u'(c)=\infty$.

## III. Competitive Equilibrium: Nonmonetary and Monetary

As in the earlier studies by Townsend (1980) and Manuelli and Sargent (1992), attention is confined here to symmetric equilibria, meaning those in which agents with identical endowments and (in the monetary case) identical money holdings receive identical consumption allocations. Also, just as the earlier studies focus on equilibria with periodic outcomes reflecting periodic endowment specifications, the analysis here is focused on equilibria in which outcomes are described as time-invariant functions of a minimal number of state variables, reflecting the simple structure of the stochastic endowment process. Under these additional assumptions, competitive equilibria for the modified turnpike economy may be completely characterized by describing the optimizing behavior of two representative agents, one of each type, and the market clearing process at a single representative trading post.

*A. Nonmonetary Equilibrium*

Itineraries are such that two agents of different types will never meet more than once along the turnpike. Moreover, there will never be a third agent who meets the first two at different times. Consequently, private debt contracts cannot be extended beyond a single N-period trading session in this environment. Each agent's expenditures and receipts must balance over the course of the N periods. This constraint on trade, as well as the time-separability of preferences, the iid structure of shocks, and the absence of storage possibilities implies that the infinite horizon economy without money may be studied as a sequence of N-period economies in which agents trade securities at each date $t \in T=\{1,N+1,2N+1,\ldots\}$ for state-contingent consumption over the following N-1 periods, taking the realized history of shocks $s^t=\{s_1,s_2,\ldots,s_t\}$ as given (since $s^t$ is revealed before agents arrive at their new posts at the beginning of the time t trading session).

Thus, for any $t=1,2,3,\ldots$, let $y^i(s^t)$ and $c^i(s^t)$ denote the endowment and consumption of a representative type i agent in period t after the history $s^t$ has been realized. Since the endowment process is iid, $y^i(s^t)=y^i(s_t)$, with $y^A(a)=y^B(b)=1$ and $y^A(b)=y^B(a)=0$.

For any $t=1,2,3,\ldots$, let $\mathbf{h}^t$ denote the set of all possible histories $s^t$ through time t. For any $t \in T$, $s^t \in \mathbf{h}^t$, and $j=0,1,\ldots,N-1$, let $\mathbf{h}^j(s^t)$ denote the set of all histories $s^{t+j}=\{s_1,\ldots,s_t,\ldots,s_{t+j}\}$ that can possibly follow $s^t$. For any $s^{t+j} \in \mathbf{h}^j(s^t)$, let $q_t(s^{t+j})$ denote the price, in the time t securities market following the realization of $s^t$, of a claim to one unit of consumption in time t+j upon the realization of $s^{t+j}$, in terms of time t consumption. Note that by definition, $q_t(s^t)=1$.

During each trading session, each agent chooses a state-contingent consumption plan that maximizes his utility subject to the requirement that his expenditures and receipts balance over the course of the N periods. That is, at each date $t \in T$, the representative type i agent solves

*Problem 1:* Given $s^t \in h^t$ and given $y^i(s^{t+j})$ and $q_t(s^{t+j})$ for all $j=0,1,\ldots,N-1$ and $s^{t+j} \in h^j(s^t)$, choose $c^i(s^{t+j})$ for all $j=0,1,\ldots,N-1$ and $s^{t+j} \in h^j(s^t)$ to maximize

$$\sum_{j=0}^{N-1} \sum_{s^{t+j} \in h^j(s^t)} (\beta/2)^j u[c^i(s^{t+j})]$$

subject to

$$\sum_{j=0}^{N-1} \sum_{s^{t+j} \in h^j(s^t)} q_t(s^{t+j}) y^i(s^{t+j}) \geq \sum_{j=0}^{N-1} \sum_{s^{t+j} \in h^j(s^t)} q_t(s^{t+j}) c^i(s^{t+j}).$$

The objective function in problem 1 indicates that the probability that any history $s^{t+j} \in h^j(s^t)$ will follow $s^t$ is just $(1/2)^j$. To repeat, the budget constraint indicates that because the spatial organization of trade rules out the use of private credit agreements across trading sessions, expenditures and receipts must balance over the course of the N periods. As N approaches infinity, the constraint becomes increasingly similar to the lifetime budget constraint faced by agents in an Arrow-Debreu economy.

The interpretation of the infinite horizon economy as a sequence of N-period economies is also exploited in defining a nonmonetary competitive equilibrium. In a nonmonetary competitive equilibrium, each agent solves problem 1 at each date $t \in T$, and markets clear at every date and state:

*Definition 1:* A *nonmonetary competitive equilibrium* consists of prices

$q_t(s^{t+j})$ and quantities $y^i(s^{t+j})$ and $c^i(s^{t+j})$ for i=A and i=B, all $t \in \mathbf{T}$, $s^t \in h^t$, j=0,1,...,N-1, and $s^{t+j} \in h^j(s^t)$ such that

(i) For i=A and i=B, the $c^i(s^{t+j})$ solve problem 1 given the $q_t(s^{t+j})$ and $y^i(s^{t+j})$.

(ii) The $y^i(s^{t+j})$ and $c^i(s^{t+j})$ satisfy

$$c^A(s^{t+j}) + c^B(s^{t+j}) = y^A(s^{t+j}) + y^B(s^{t+j}) = 1.$$

The market clearing condition listed as (ii) in definition 1 reflects the fact that there are equal numbers of agents of each type at each trading post, so that the aggregate endowment at the representative trading post can be normalized to unity.

The representative type i agent's first order conditions for problem 1 are given by

$$(1) \quad (\beta/2)^j u'[c^i(s^{t+j})] = q_t(s^{t+j})u'[c^i(s^t)],$$

for all j=0,1,...,N-1 and $s^{t+j} \in h^j(s^t)$.

Equation (1), along with the market clearing condition from definition 1, implies that $q_t(s^{t+j})=(\beta/2)^j$ and

$$(2) \quad c^i(s^{t+j}) = c^i(s^t),$$

for all j=0,1,...,N-1 and $s^{t+j} \in h^j(s^t)$.

Thus, for each agent, consumption is constant across all date-state combinations within a given trading session. The budget constraint from problem 1 pins down the constant level of consumption as

$$(3) \quad c^i(s^t) = [(1-\beta)y^i(s_t)+(\beta/2)(1-\beta^{N-1})]/(1-\beta^N).$$

It is useful to compare the equilibrium allocation described by (2) and (3) to the Pareto optimal allocation under which all agents are treated alike. This allocation would be supported, for example, in the competitive equilibrium of an economy having the same endowment specification as the turnpike model but in which the spatial constraints on exchange are relaxed to permit agents to trade in a centralized Arrow-Debreu securities market. The Pareto optimal allocation maximizes the sum of the two representative agents' expected utilities subject to an aggregate resource constraint for each date-state combination. Thus, it may be characterized by solving

*Problem 2:* Choose $c^A(s^t)$ and $c^B(s^t)$ for all $t=1,2,3,\ldots$ and $s^t \in \mathbf{h}^t$ to maximize

$$\sum_{t=1}^{\infty} \sum_{s^t \in \mathbf{h}^t} (\beta/2)^t u[c^A(s^t)] + \sum_{t=1}^{\infty} \sum_{s^t \in \mathbf{h}^t} (\beta/2)^t u[c^B(s^t)]$$

subject to

$$1 \geq c^A(s^t) + c^B(s^t) \qquad \text{for all } t=1,2,3,\ldots \text{ and } s^t \in \mathbf{h}^t.$$

The solution to problem 2 sets $c^A(s^t)=c^B(s^t)=1/2$ for all $t=1,2,3,\ldots$ and $s^t \in \mathbf{h}^t$. In contrast to the optimal allocation, which equates each agent's marginal utilities across all date-state combinations, equilibrium consumptions depend on the realization of the time t shock $s_t$, and hence marginal utilities are equated only during those periods in which groups of agents remain together at a trading post. In equilibrium when N=1, equation (3) reveals that agents must simply consume their own endowments in every period. The exchange of private securities supports more trade as N grows larger, but even when $N=\infty$ the optimum is not achieved since agents cannot trade before the realization of $s_1$.

*B. Monetary Equilibrium*

Noninterest-bearing government-issued money may be introduced into this economy via lump-sum transfers of $m_1^i$ to each type i agent just prior to the initial period. The constant size of the aggregate money supply is normalized so that $m_1^A+m_1^B=1$. The potential value of money in this economy rests on its status as the only available outside asset. Unlike private securities that must be redeemed before agents move on, money can be carried across trading sessions and thereby permits otherwise infeasible transactions to occur between strangers.

While the infinite horizon monetary economy may still be studied as a sequence of N-period economies with state-contingent securities markets open at each date $t \in \mathbf{T}$, now each member of the sequence is linked to its predecessor and its follower by the distribution of cash balances across agent types at the beginning and end of each trading session. In the time t securities market, agents trade claims for state-contingent consumption over the following N-1 periods and claims for state-contingent delivery in period t+N-1 of money to be carried into period t+N.

Extending the notational conventions established above, let $m^i(s^{t-1})$ denote the nominal balances held at the end of period t-1, and hence carried into period t, by the representative type i agent after the history $s^{t-1}$ has been realized. For $t \in \mathbf{T}$, $s^t \in \mathbf{h}^t$, and $j=0,1,\ldots,N-1$, let $p_t(s^{t+j})$ be the price, in the time t securities market following the realization of $s^t$, of a claim to one unit of money in time t+j upon the realization of $s^{t+j} \in \mathbf{h}^j(s^t)$, in terms of time t consumption. Note that $p_t(s^t)$ is just the inverse of the price level at time t.

As sources of funds during the time t trading session, an agent has the money that he carries into the trading session as well as the receipts from the sale of his endowment. As uses of funds, an agent has the money that he carries out of the trading session as well as his expenditures on consumption. During each trading session, an agent chooses a state-contingent consumption and money-holdings plan that maximizes his utility subject to the constraint that his sources and uses of funds balance over the course of the N periods. Thus, the representative type i agent solves

*Problem 3:* Given $m_1^i$ and given $y^i(s^{t+j})$, $q_t(s^{t+j})$, and $p_t(s^{t+j})$ for all $t \in T$, $s^t \in h^t$, $j=0,1,\ldots,N-1$, and $s^{t+j} \in h^j(s^t)$, choose $c^i(s^{t+j})$ and $m^i(s^{t+N-1})$ for all $t \in T$, $s^t \in h^t$, $j=0,1,\ldots,N-1$, $s^{t+j} \in h^j(s^t)$, and $s^{t+N-1} \in h^{N-1}(s^t)$ to maximize

$$\sum_{t \in T} \sum_{s^t \in h^t} \sum_{j=0}^{N-1} \sum_{s^{t+j} \in h^j(s^t)} (\beta/2)^{t+j} u[c^i(s^{t+j})]$$

subject to

$$p_t(s^t)m^i(s^{t-1}) + \sum_{j=0}^{N-1} \sum_{s^{t+j} \in h^j(s^t)} q_t(s^{t+j})y^i(s^{t+j})$$

$$\geq \sum_{j=0}^{N-1} \sum_{s^{t+j} \in h^j(s^t)} q_t(s^{t+j})c^i(s^{t+j}) + \sum_{s^{t+N-1} \in h^{N-1}(s^t)} p_t(s^{t+N-1})m^i(s^{t+N-1}).$$

for all $t \in T$ and $s^t \in h^t$

and

$$m^i(s^{t+N-1}) \geq 0 \qquad \text{for all } t \in T,\ s^t \in h^t, \text{ and } s^{t+N-1} \in h^{N-1}(s^t).$$

The objective function in problem 3 indicates that the probability that any history $s^{t+j}$ will be realized through time t+j is just $(1/2)^{t+j}$. The budget constraints indicate that, while expenditures and receipts must still balance over the course of the N-period trading session, the availability of valued money now permits agents to carry purchasing power across trading sessions. In the modified turnpike environment with N=1, as in the pure monetary economies studied by Gale (1978) and Townsend (1980), money is used as a means of exchange in every period. As N gets larger, the constraints reveal that money will change hands less frequently.

A monetary equilibrium is defined as a set of prices and quantities such that each agent is solving problem 3 and such that all markets clear:

*Definition 2:* A *monetary competitive equilibrium* consists of the initial conditions $m_1^A$ and $m_1^B$, the prices $q_t(s^{t+j})$ and $p_t(s^{t+j})$, and the quantities $y^i(s^{t+j})$, $c^i(s^{t+j})$, and $m^i(s^{t+N-1})$ for i=A and i=B, all $t\in\mathbf{T}$, $s^t\in\mathbf{h}^t$, $j=0,1,\ldots,N-1$, $s^{t+j}\in\mathbf{h}^j(s^t)$, and $s^{t+N-1}\in\mathbf{h}^{N-1}(s^t)$ such that

(i) For i=A and i=B, the $c^i(s^{t+j})$ and $m^i(s^{t+N-1})$ solve problem 3 given the $m_1^i$, $q_t(s^{t+j})$, $p_t(s^{t+j})$, and $y^i(s^{t+j})$.

(ii) The $y^i(s^{t+j})$, $c^i(s^{t+j})$, and $m^i(s^{t+N-1})$ satisfy

$$c^A(s^{t+j}) + c^B(s^{t+j}) = y^A(s^{t+j}) + y^B(s^{t+j}) = 1$$

and

$$m^A(s^{t+N-1}) + m^B(s^{t+N-1}) = 1.$$

As before, the market clearing conditions in definition 3 reflect the fact that the aggregate endowment and money supply are normalized to unity.

The type i representative agent's first order conditions for problem 3 are

$$(4) \quad (\beta/2)^j u'[c^i(s^{t+j})] = q_t(s^{t+j})u'[c^i(s^t)],$$

for all $t\in\mathbf{T}$, $s^t\in\mathbf{h}^t$, $j=0,1,\ldots,N-1$, and $s^{t+j}\in\mathbf{h}^j(s^t)$

and

$$(5) \quad p_t(s^{t+N-1})u'[c^i(s^t)] \;\geq\; E_{t+N-1}\left\{ (\beta^N/2^{N-1})p_{t+N}(s^{t+N})u'[c^i(s^{t+N})] \right\},$$

with equality if $m^i(s^{t+N-1})>0$, for all $t\in\mathbf{T}$, $s^t\in\mathbf{h}^t$, and $s^{t+N-1}\in\mathbf{h}^{N-1}(s^t)$,

where $E_{t+N-1}\{\cdot\}$ indicates that the expectation is conditional on time t+N-1 information, that is, conditional on the realization of $s^{t+N-1}$.

As in the nonmonetary case, equation (4) along with the market clearing conditions for goods implies that $q_t(s^{t+j})=(\beta/2)^j$ and

$$(6) \quad c^i(s^{t+j}) \;=\; c^i(s^t),$$

for all $t\in\mathbf{T}$, $s^t\in\mathbf{h}^t$, $j=0,1,\ldots,N-1$, and $s^{t+j}\in\mathbf{h}^j(s^t)$.

The absence of arbitrage opportunities in the money market guarantees that

$$(7) \quad p_t(s^t) \;=\; \sum_{s^{t+N-1}\in\mathbf{h}^{N-1}(s^t)} p_t(s^{t+N-1})$$

will hold for all $t\in\mathbf{T}$ and $s^t\in\mathbf{h}^t$. Otherwise, any agent could buy (sell) nominal balances at time t after $s^t$ has been realized, simultaneously sell (buy) claims to equal quantities of nominal balances for each $s^{t+N-1}\in\mathbf{h}^{N-1}(s^t)$, and thereby make a certain profit. Summing (5) over all $s^{t+N-1}\in\mathbf{h}^{N-1}(s^t)$ and using (7) yields a stochastic Euler equation of the form

$$(8) \quad p_t(s^t)u'[c^i(s^t)] \;\geq\; E_t\left\{ \beta^N p_{t+N}(s^{t+N})u'[c^i(s^{t+N})] \right\},$$

which holds with equality whenever $m^i(s^{t+N-1})>0$ for all $s^{t+N-1}\in\mathbf{h}^{N-1}(s^t)$.

Equations (2) and (6) reveal that in both nonmonetary and monetary equilibria, marginal utilities are equated across periods, and hence the gains from trade are exhausted, within each trading session. Equation (8) summarizes the extent to which the availability of outside money permits additional welfare-improving exchange across sessions as trading partners change. The nonnegativity constraint on money holdings makes this inter-session Euler equation an inequality. As emphasized by Zeldes (1989), even if the nonnegativity constraint does not bind in the current period, the possibility that it will bind in some future period is enough to lower consumption today. That is, even when it holds with equality, equation (8) indicates that the gains from trade are only partially exhausted across trading sessions. Thus, government-issued money is a valuable, but imperfect, means of exchange between agents who are meeting as strangers.

No arbitrage conditions may be used to deduce what within-session paths for prices and money holdings are if trading in money, goods, and one-period-ahead contingent claims takes place at each date t=1,2,3,.... For $t \in \mathbf{T}$ and $j=1,2,\ldots,N-1$, let $p_{t+j}(s^{t+j})$ be the spot price of money in terms of goods in period t+j when history $s^{t+j}$ has been realized. Since agents have no reason to prefer money to privately-issued securities within a given trading session, cash must yield the same return as is available from any other asset between t and t+j; the Euler equation

$$p_t(s^t)u'[c^i(s^t)] \quad = \quad \beta^j p_{t+j}(s^{t+j})u'[c^i(s^{t+j})]$$

must hold in equilibrium. Equation (6) then implies that $p_t(s^t)=\beta^j p_{t+j}(s^{t+j})$; deflation prevails within each session.

Also because private securities can substitute perfectly for money within a trading session, it may be assumed without loss of generality that $m^i(s^{t+j-1})=m^i(s^{t-1})$ for all $t \in \mathbf{T}$, all $j=0,1,\ldots,N-1$ and all $s^{t+j-1}$ that follow $s^{t-1}$. That is, money only changes hands at the end of each trading session. Money in the modified turnpike model is used exclusively to facilitate trade across trading sessions; within trading sessions, private securities do all the work.

## IV. The Effects of Lengthening the Trading Session

It is easy, using equations (2) and (3), to numerically compute a nonmonetary equilibrium for the model economy with trading sessions of length *N*. Once a realization for $\{s_t\}_{t=1}^{\infty}$ is drawn, the implied values for the $y^i(s_t)$ can be plugged into (3) to determine the $c^i(s^t)$, which by (2) completely describe consumptions at each date-state combination within each trading session. Finding monetary equilibria is a more difficult task, however, since the $c^i(s^t)$, $m^i(s^{t+N-1})$, and $p_t(s^t)$ must be constructed to satisfy the stochastic Euler equation given by (8) for both i=A and i=B.

For $t \in T$, define

$$\xi_t = \begin{cases} 1 \text{ if } s_t=a \\ 0 \text{ if } s_t=b \end{cases} . \tag{9}$$

The symmetry and stationarity assumptions made in the previous section justify the conjecture that because the time t division of the money supply and the time t realization of the endowment shock are the only variables that distinguish the time t trading session from any other, equilibrium values for $c^i(s^t)$, $m^i(s^{t+N-1})$, and $p_t(s^t)$ may be expressed as time-invariant functions of $m_t=m^A(s^{t-1})=1-m^B(s^{t-1})$ and $\xi_t$, so that

$$\begin{aligned} c^A(s^t) &= 1 - c^B(s^t) = c(m_t,\xi_t) \\ m_{t+N} &= m^A(s^{t+N-1}) = 1 - m^B(s^{t+N-1}) = \mu(m_t,\xi_t) \\ p_t(s^t) &= p(m_t,\xi_t). \end{aligned} \tag{10}$$

The functions $c(m,\xi)$, $\mu(m,\xi)$, and $p(m,\xi)$ that satisfy equations (8)-

(10) may be solved for using the numerical procedures outlined in the appendix. Once these functions have been found, the initial condition $m_1=m_1^A$ and a sample realization for $\{s_t\}_{t=1}^{\infty}$ may be fed into equations (9) and (10) to generate the equilibrium values for $c^i(s^t)$, $m^i(s^{t+N-1})$, and $p_t(s^t)$.

In particular, the function $\mu(m,\xi)$ governs how the distribution of money balances across agent types evolves over time. For $t \in T$, let $\lambda_t(m)$ denote the probability, prior to the initial period, that type A agents will each carry m units of money into the time t trading session. In each of the examples considered below, the function $\mu(m,\xi)$ is such that $\lambda_t(m)$ converges to a limiting distribution as t approaches infinity, regardless of the initial conditions $m_1^i$, just as in Foley and Hellwig (1975). That is, in each case the economy reaches a stochastic steady state in which prices and quantities are strictly stationary random variables.

The numerical procedures are implemented to construct nonmonetary and monetary equilibria for values of N ranging from 1 through 6. The utility function is specialized to the CES form,

$$u(c) \quad = \quad (c^{1-\sigma}-1)/(1-\sigma), \qquad \sigma>0.$$

In order to focus on the effects of changing N, the preference parameters are held fixed, with $\beta$=0.95 and $\sigma$=2.

Figures 1-6 display the functions $c(m,\xi)$ and $\mu(m,\xi)$ for the monetary equilibria. In each case, c and $\mu$ are increasing in both of their arguments (except for minor nonmonotonicities that are artifacts of the numerical approximation procedure). An agent with larger cash holdings consumes more than an agent with less money. An agent who receives an endowment at the beginning of a trading session both consumes more and accumulates money relative to an agent who receives no endowment. Since additional trade becomes possible as N increases, the function $c(m,\xi)$ is closer to being constant at the Pareto optimal level of 1/2 in the economies with larger N than in those with smaller N. Also, the constraints $1\geq\mu$ and $\mu\geq0$ are more likely to be binding as N increases; as the scope for private debt expands, agents no longer need to constantly hold precautionary balances.

With both the aggregate endowment and the money supply fixed at unity, the equation of exchange MV=PY reduces to V=P, indicating that the income velocity of money is exactly equal to the price level in the modified turnpike economies. Since the price level at time $t\in\mathbf{T}$ in state $s^t\in\mathbf{h}^t$ is just $1/p_t(s^t)$, velocity is described by the time-invariant function $v(m,\xi)=1/p(m,\xi)$. Table 1 reports selected moments for the price level/velocity process and a representative agent's endowment and consumption process in each example economy's stochastic steady state.

An agent's endowment is either 0 or 1 with equal probability, so the variance of this process is 0.25 in every case. In both nonmonetary and monetary equilibria, the variance of consumption decreases monotonically as N increases, reflecting the additional trade that becomes possible with longer trading sessions. The gains are largest moving from N=1 to N=2 and from N=2 to N=3. As N increases, equilibrium consumption patterns approach those provided by Pareto optimal allocations, which eliminate all variability in individual agents' consumption streams.

Comparing the variance of consumption in nonmonetary and monetary equilibria for any fixed N reveals the extent to which the availability of an outside asset expands opportunities for exchange. Money is especially important in economies with low values of N, where agents are frequently meeting as strangers. Hence, the price level and the velocity of money are low in these equilibria and increase monotonically with N.

The additional welfare-improving exchange made possible by lengthening the trading session and introducing money is also illustrated in table 2, which reports a representative agent's expected utility in the stochastic steady state of each example economy. The welfare gain is most dramatic when money is introduced into the economy with N=1, since the nonmonetary equilibrium there is autarkic. As suggested by the comparison of consumption patterns, expected utility in nonmonetary equilibria, monetary equilibria, and under Pareto optimal allocations converge as N increases.

## V. Conclusions and Directions for Future Research

Among the economies studied here, the stochastic turnpike model with N=1 most closely resembles Townsend's (1980) original turnpike environment. Agents in this economy continually meet as strangers at isolated trading posts. Markets for private securities are ruled out completely. Government-issued money is critical in facilitating trade. Monetary equilibria, though far superior to nonmonetary equilibria, are generally not Pareto optimal.

In stark contrast to the N=1 economy is one in which the spatial constraints on exchange are relaxed to permit agents to trade a full set of Arrow-Debreu securities in a single centralized market. There are no strangers in this alternative environment. The gains from trade are completely exhausted through the transfer of privately-issued financial assets. There is no need for money. Competitive equilibria are Pareto optimal.

The results presented in section IV show how the stochastic turnpike economies with $N \geq 2$ represent cases intermediate to the pure monetary environment on the one hand and the Arrow-Debreu environment on the other. Agents encounter strangers less frequently as N increases. More trade becomes possible through the exchange of private securities. Money becomes less valuable, so that the price level and velocity both rise. Welfare increases, approaching the level achieved by Pareto optimal allocations. All of these changes proceed smoothly as the trading session is lengthened, making N an exact index measuring distances between the two extreme cases.

In actual economies, payments systems usually feature both privately-issued and government-issued assets as means of exchange. The model economies presented here capture this reality; the original turnpike model and the Arrow-Debreu framework do not. The results from section IV imply that asset velocities ought to vary to the extent that decentralized trade among strangers becomes more or less common, both across countries at any given point in time and over time within any given country. Evidence consistent with this implication is found, in fact, by Townsend (1983) and Bordo and Jonung (1987).

Since the analysis in this paper takes the nominal money supply as fixed, two issues are left for future research. The first concerns the effects of monetary injections, either once-and-for-all or ongoing. If, following Scheinkman and Weiss (1986), preferences are respecified to include leisure in the utility function and a technology is specified so that $s_t$ summarizes idiosyncratic productivity shocks instead of endowment shocks, then monetary policies that change the cross-sectional distribution of money holdings will have real aggregate effects. It will be more difficult, however, for monetary policy to generate significant distributive effects when real balances represent a smaller fraction of each agent's total asset holdings. Thus, the magnitude and duration of the nonneutralities--the efficacy of monetary policy--is likely to diminish as N increases.

Once the effects of arbitrary monetary policies have been analyzed, the next question concerns what constitutes the optimal monetary policy in this environment. The answer is likely to depend sensitively on the informational constraints that the government is assumed to face: whether it can observe the realization of the endowment shock, whether it can keep track of histories of shocks, and whether it can observe individual agents' asset holdings. Work by Levine (1991), which also uses a model in which agents have a demand for precautionary balances, suggests that if the government is constrained to treat all agents alike, the traditional result of Friedman (1969) calling for a steady contraction of the money supply may be overturned in favor of an inflationary policy that helps to prevent unlucky agents from running out of cash.

## Appendix: Numerical Methods

Monetary equilibria for the stochastic turnpike economy are constructed numerically as solutions to a system of functional equations. Substitute the functions defined by equation (10) into the representative type A agent's budget constraint, which will always hold with equality, to obtain

$$\text{(A.1)} \quad p(m_t,\xi_t)m_t + \xi_t + \frac{\beta(1-\beta^{N-1})}{2(1-\beta)} = \frac{(1-\beta^N)c(m_t,\xi_t)}{1-\beta} + p(m_t,\xi_t)\mu(m_t,\xi_t),$$

where equations (6), (7), and (9) and the result that $q_t(s^{t+j})=(\beta/2)^j$ have all been used. Substitute the functions into the stochastic Euler equation given by (8) to obtain

$$\text{(A.2)} \quad p(m_t,\xi_t)u'[c(m_t,\xi_t)] \geq E_t\left\{ \beta^N p(\mu(m_t,\xi_t),\xi_{t+N})u'[c(\mu(m_t,\xi_t),\xi_{t+N})] \right\}$$

and

$$\text{(A.3)} \quad p(m_t,\xi_t)u'[1-c(m_t,\xi_t)] \geq E_t\left\{ \beta^N p(\mu(m_t,\xi_t),\xi_{t+N})u'[1-c(\mu(m_t,\xi_t),\xi_{t+N})] \right\}.$$

Since (A.1)-(A.3) must hold for all $m_t \in [0,1]$ and $\xi_t \in \{0,1\}$, they may be rewritten as the functional equations

$$\text{(A.4)} \quad p(m,\xi)m + \xi + \frac{\beta(1-\beta^{N-1})}{2(1-\beta)} = \frac{(1-\beta^N)c(m,\xi)}{1-\beta} + p(m,\xi)\mu(m,\xi),$$

$$\text{(A.5)} \quad p(m,\xi)u'[c(m,\xi)] - \gamma(m,\xi) = \frac{\beta^N}{2}\left[ p(\mu(m,\xi),1)u'[c(\mu(m,\xi),1)] + p(\mu(m,\xi),0)u'[c(\mu(m,\xi),0)] \right],$$

and

$$\text{(A.6)} \quad p(m,\xi)u'[1-c(m,\xi)] - \varphi(m,\xi)$$

$$= \frac{\beta^N}{2} \Big[ p(\mu(m,\xi),1)u'[1-c(\mu(m,\xi),1)] + p(\mu(m,\xi),0)u'[1-c(\mu(m,\xi),0)] \Big] ,$$

where

(A.7) $\gamma(m,\xi)\mu(m,\xi) = 0, \qquad \gamma(m,\xi)\geq 0,$

and

(A.8) $\varphi(m,\xi)[1-\mu(m,\xi)] = 0, \qquad \varphi(m,\xi)\geq 0.$

In addition, the symmetry assumption requires that

(A.9) $c(m,\xi) = 1 - c(1-m,1-\xi),$

(A.10) $\mu(m,\xi) = 1 - \mu(1-m,1-\xi),$

(A.11) $p(m,\xi) = p(1-m,1-\xi),$

and

(A.12) $\gamma(m,\xi) = \varphi(1-m,1-\xi).$

Solutions to the system (A.4)-(A.12) are found numerically by modifying the spline approximation technique discussed in Hildebrand (1974, Ch.9). The domain [0,1] is divided into five intervals of equal length. The functions $\mu(m,0)$ and $p(m,0)$ are approximated by a cubic polynomial on each interval:

(A.13)
$$\mu(m,0) \approx a_k + b_k m + c_k m^2 + d_k m^3$$
$$p(m,0) \approx e_k + f_k m + g_k m^2 + h_k m^3$$

where k=1 if $m\in[0,0.2)$, k=2 if $m\in[0.2,0.4)$, k=3 if $m\in[0.4,0.6)$, k=4 if $m\in[0.6,0.8)$, and k=5 if $m\in[0.8,1]$. Given these approximations for $\mu(m,0)$ and $p(m,0)$, approximations for $c(m,0)$, $\mu(m,1)$, $p(m,1)$, and $c(m,1)$ are constructed so that equations (A.4) and (A.9)-(A.11) are satisfied.

The 40 unknown coefficients $\{a_k,b_k,c_k,d_k,e_k,f_k,g_k,h_k\}_{k=1}^{5}$ are determined by requiring that $\mu(m,0)$ and $p(m,0)$ be continuous at m=0.2, m=0.4, m=0.6, and m=0.8 (a total of 8 equations that must be satisfied) and by requiring that $\mu(m,0)$ and $p(m,0)$ satisfy (A.5) and (A.6) at 16 uniformly-spaced points on [0,1] (a total of 32 equations that must be satisfied). Approximations for $\gamma(m,0)$ and $\varphi(m,0)$ are determined so that (A.7) and (A.8) also hold at the same 16 uniformly-spaced points. Finally, approximations for $\gamma(m,1)$ and $\varphi(m,1)$ are constructed to satisfy equation (A.12). Throughout, the constraints $1>c(m,\xi)>0$, $1\geq\mu(m,\xi)\geq0$, and $p(m,\xi)>0$ for all $m\in[0,1]$ and $\xi\in\{0,1\}$ are imposed.

The limiting distribution of money holdings, $\lambda(m)$, is found by confining the approximation for $\mu(m,\xi)$ to the finite set $\mathbf{M}=\{0,0.01,0.02,\ldots,1\}$ and solving the equations

$$\lambda(m') = \frac{1}{2} \sum_{m\in\mathbf{M}_0(m')} \lambda(m) + \frac{1}{2} \sum_{m\in\mathbf{M}_1(m')} \lambda(m)$$

for all $\lambda(m')$ such that $m'\in\mathbf{M}$, where

$$\mathbf{M}_0(m') = \{\ m\in\mathbf{M} \mid \mu(m,0)=m'\ \} \quad \text{and} \quad \mathbf{M}_1(m') = \{\ m\in\mathbf{M} \mid \mu(m,1)=m'\ \}.$$

This solution procedure specializes the projection methods outlined by Judd (1992) by choosing the set of piecewise cubic functions in equation (A.13) as a basis and a collocation method for the projection conditions. Piecewise cubic approximation allows for considerable nonlinearity in the functions describing consumptions, money holdings, and prices. Imposing continuity on the functions but not (as is usually done in spline approximation) their first and second derivatives allows for kinks associated with binding nonnegativity constraints on money holdings.

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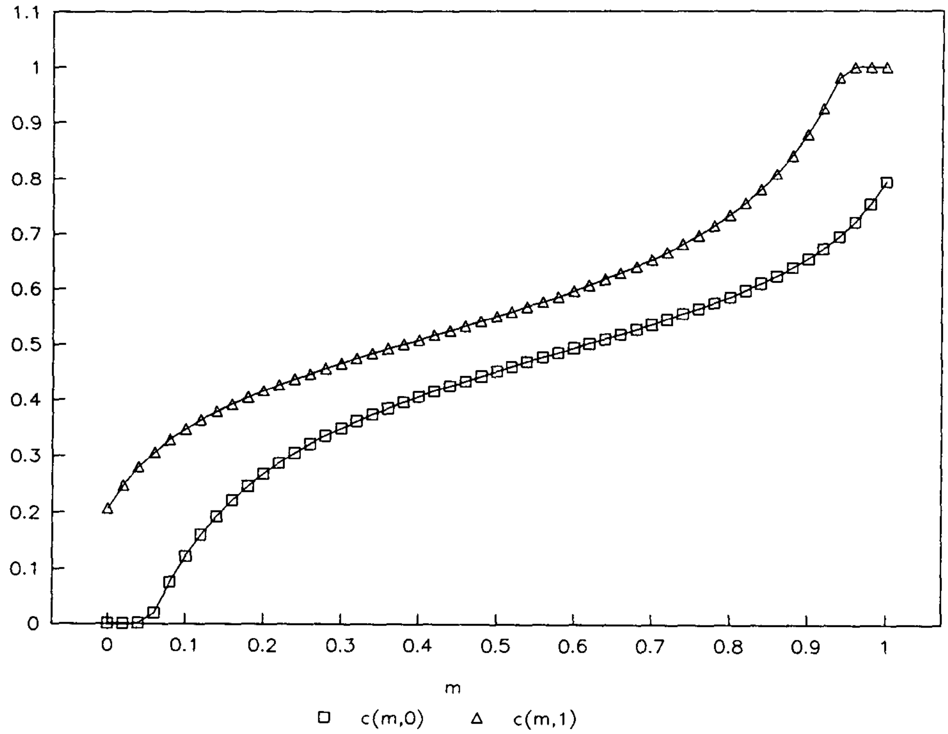
Table 1.--Moments for Endowments, Consumption, and Velocity

| $\beta$=0.95, $\sigma$=2 | N | Variance of Endowment Process | Variance of Consumption Process | Mean of Price/Velocity Process |
|---|---|---|---|---|
| Nonmonetary Equilibria | 1 | 0.2500 | 0.2500 | |
| | 2 | 0.2500 | 0.0657 | |
| | 3 | 0.2500 | 0.0307 | |
| | 4 | 0.2500 | 0.0182 | |
| | 5 | 0.2500 | 0.0122 | |
| | 6 | 0.2500 | 0.0089 | |
| Monetary Equilibria | 1 | 0.2500 | 0.0142 | 0.2176 |
| | 2 | 0.2500 | 0.0128 | 0.6472 |
| | 3 | 0.2500 | 0.0101 | 1.1004 |
| | 4 | 0.2500 | 0.0090 | 1.9817 |
| | 5 | 0.2500 | 0.0077 | 3.4450 |
| | 6 | 0.2500 | 0.0067 | 5.9124 |
| Pareto Optimum | | 0.2500 | 0.0000 | |

Table 2.--Expected Utilities

| $\beta$=0.95, $\sigma$=2<br>N | Nonmonetary Equilibria | Monetary Equilibria | Equal Weight Pareto Optimum |
|---|---|---|---|
| 1 | $-\infty$ | -22.4106 | -19.0000 |
| 2 | -32.5593 | -21.3638 | -19.0000 |
| 3 | -24.3245 | -20.6967 | -19.0000 |
| 4 | -21.9773 | -20.4450 | -19.0000 |
| 5 | -20.9517 | -20.2287 | -19.0000 |
| 6 | -20.4037 | -20.0538 | -19.0000 |

Fig. 1. N=1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0.8
0.7
0.6
0.5
0.4
0.3
0.2
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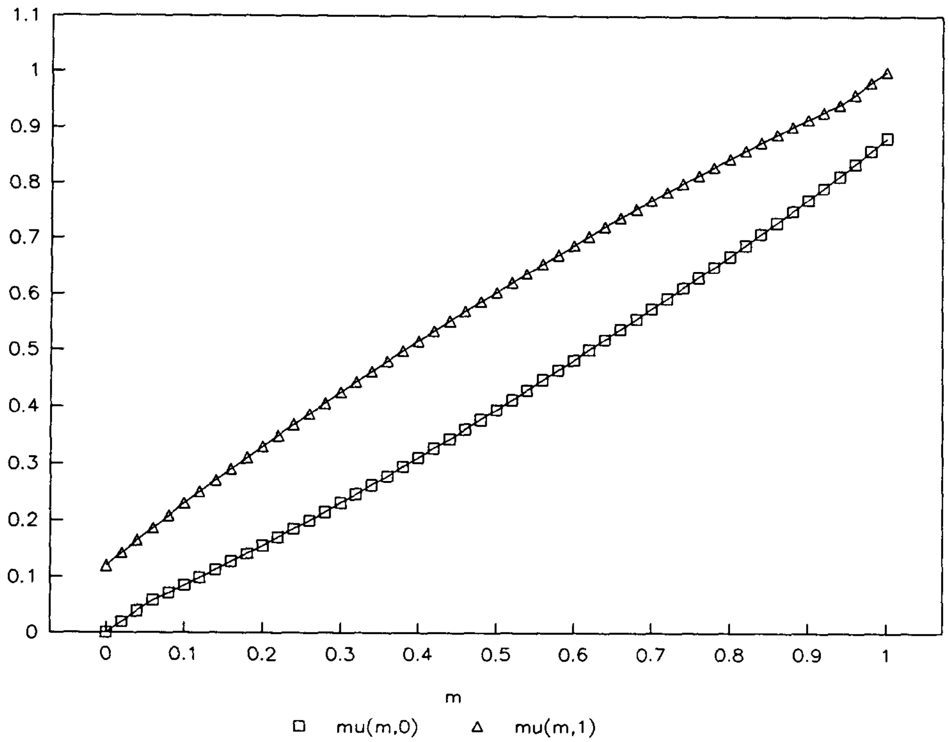
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
m
c(m,0)
c(m,1)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0.2
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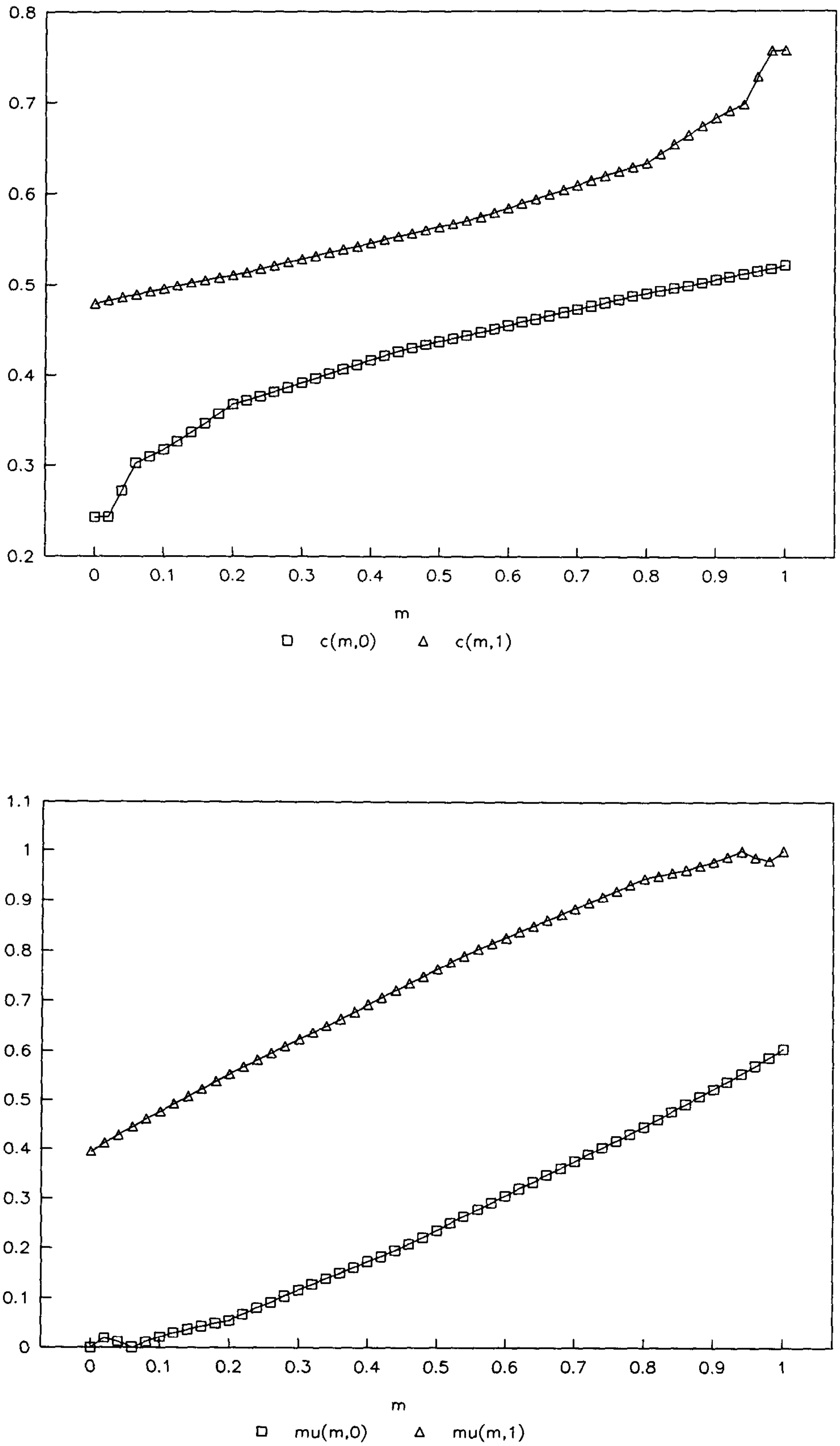
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
m
mu(m,0)
mu(m,1)

Fig. 2. N=2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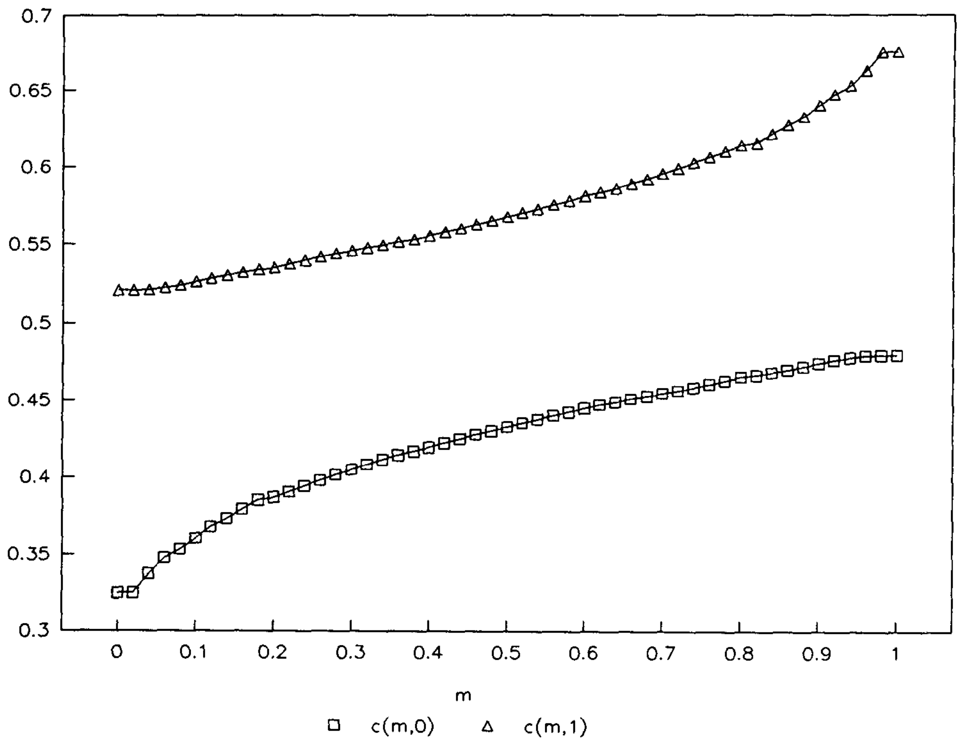
0.5
0.6
0.7
0.8
0.9
1
m
c(m,0)
c(m,1)
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
m
mu(m,0)
mu(m,1)

Fig. 3. N=3



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0.3
0
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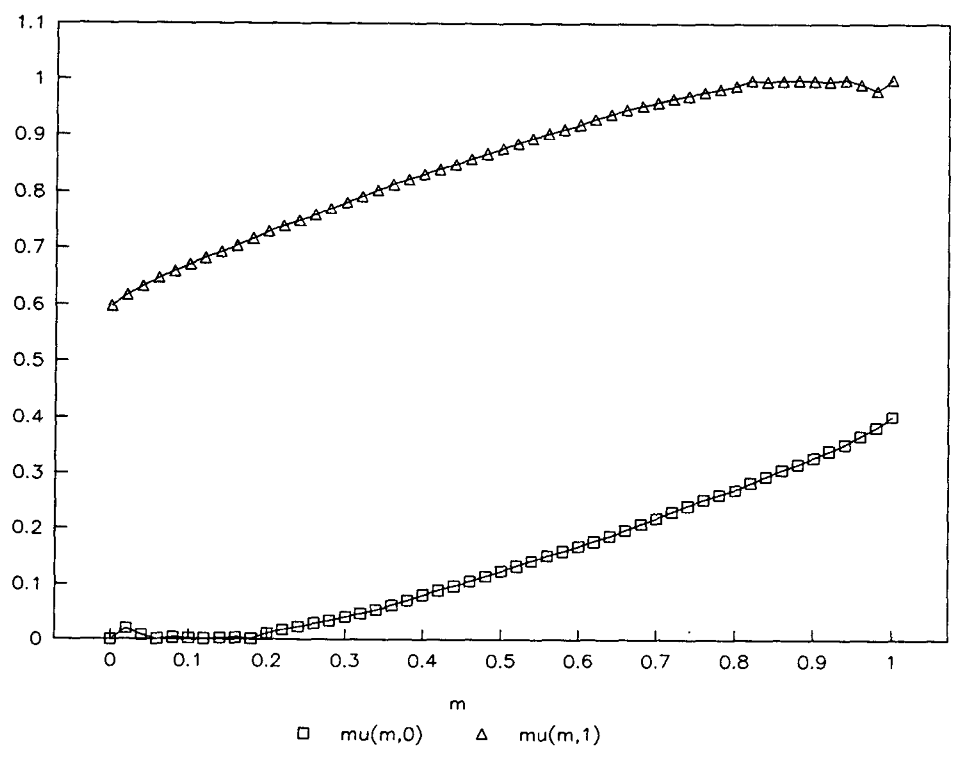
0.7
0.8
0.9
1
m
c(m,0)
c(m,1)

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0.8
0.7
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0.4
0.3
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0
0
0.1
0.2
0.3
0.4
0.5
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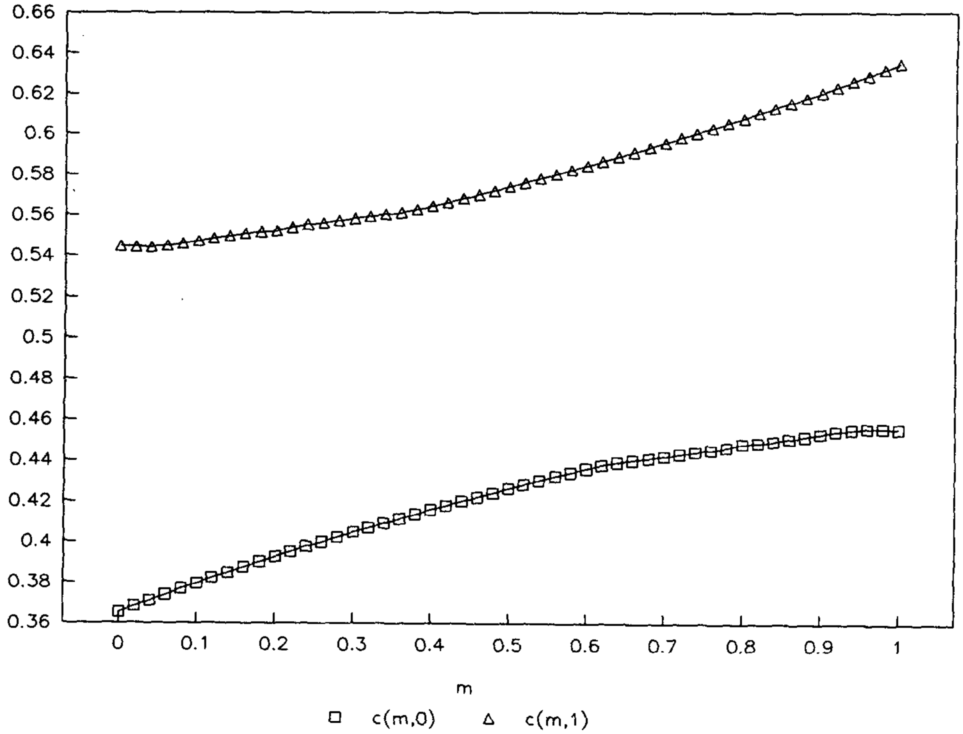
0.7
0.8
0.9
1
m
mu(m,0)
mu(m,1)

Fig. 4. N=4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0.4
0.38
0.36
0
0.1
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0.7
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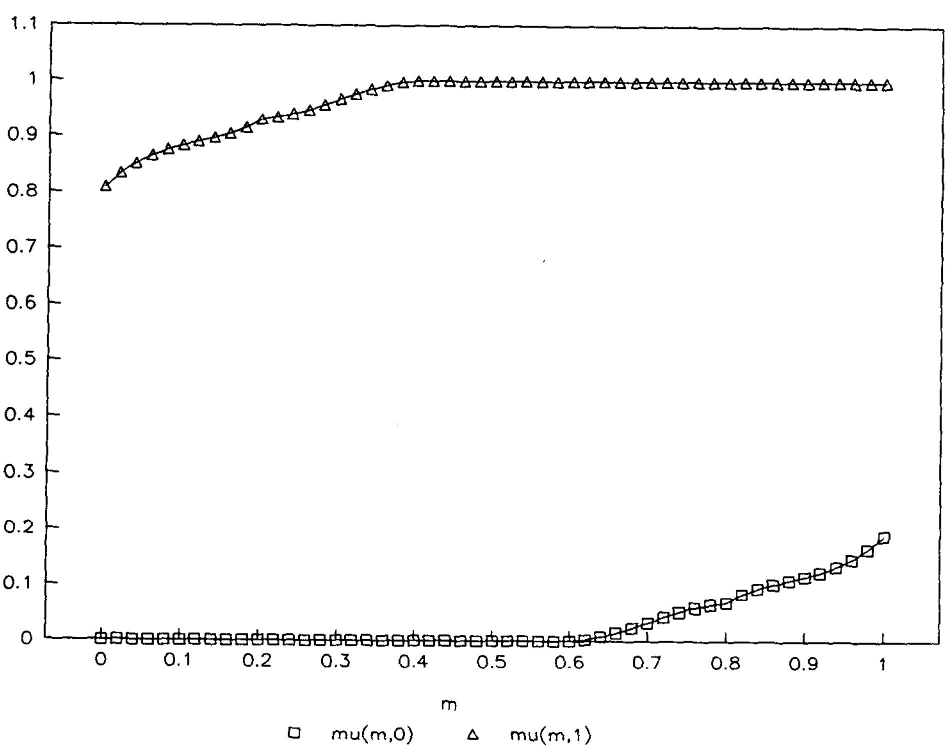
0.9
1
m
c(m,0)
c(m,1)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0.6
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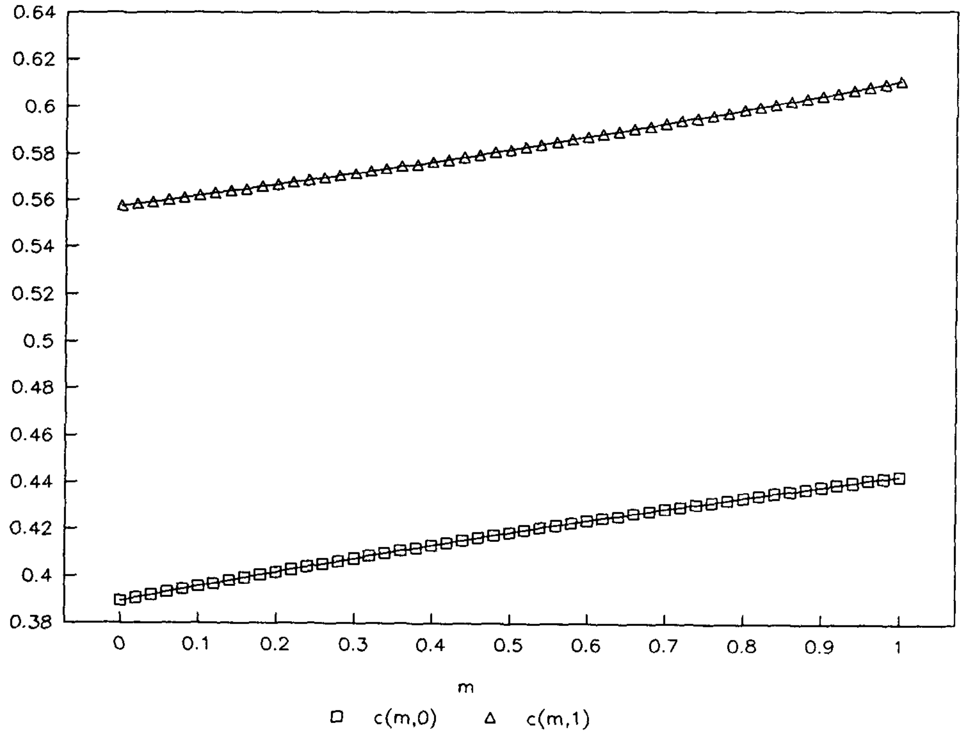
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
m
mu(m,0)
mu(m,1)

Fig. 5. N=5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0.46
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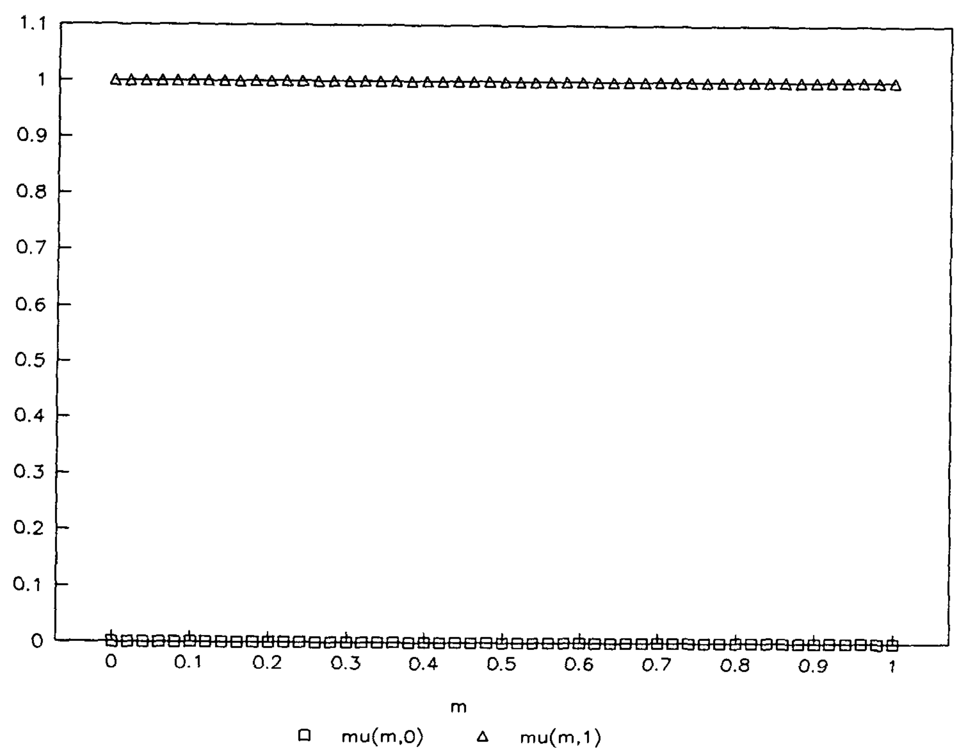
0.4
0.38
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
m
c(m,0)
c(m,1)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0.8
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0.6
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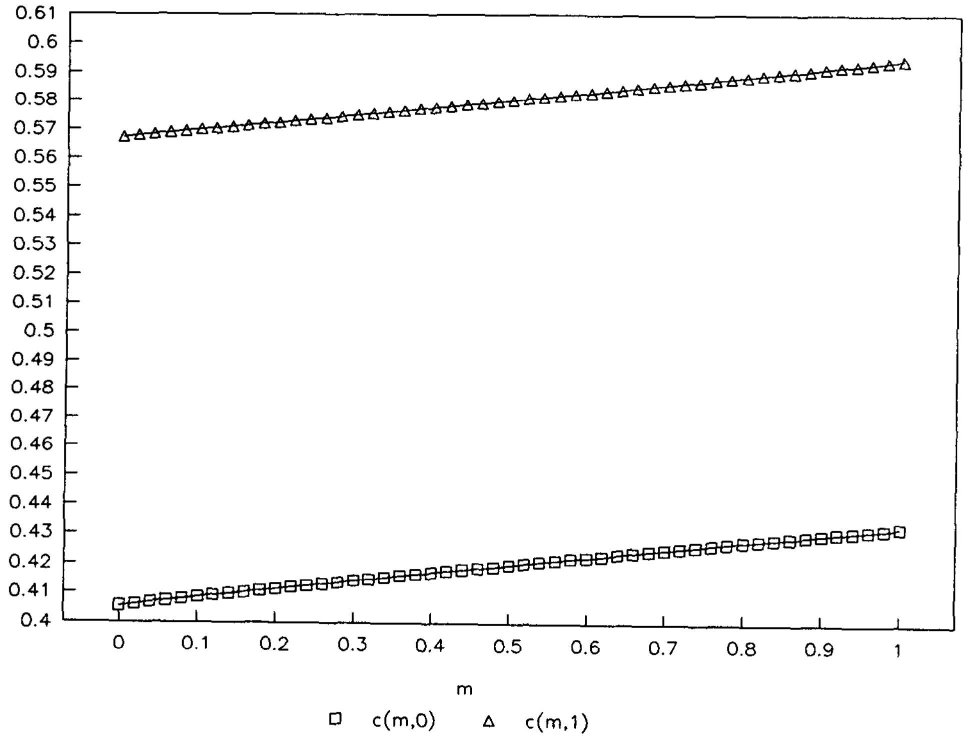
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
m
mu(m,0)
mu(m,1)

Fig. 6. N=6



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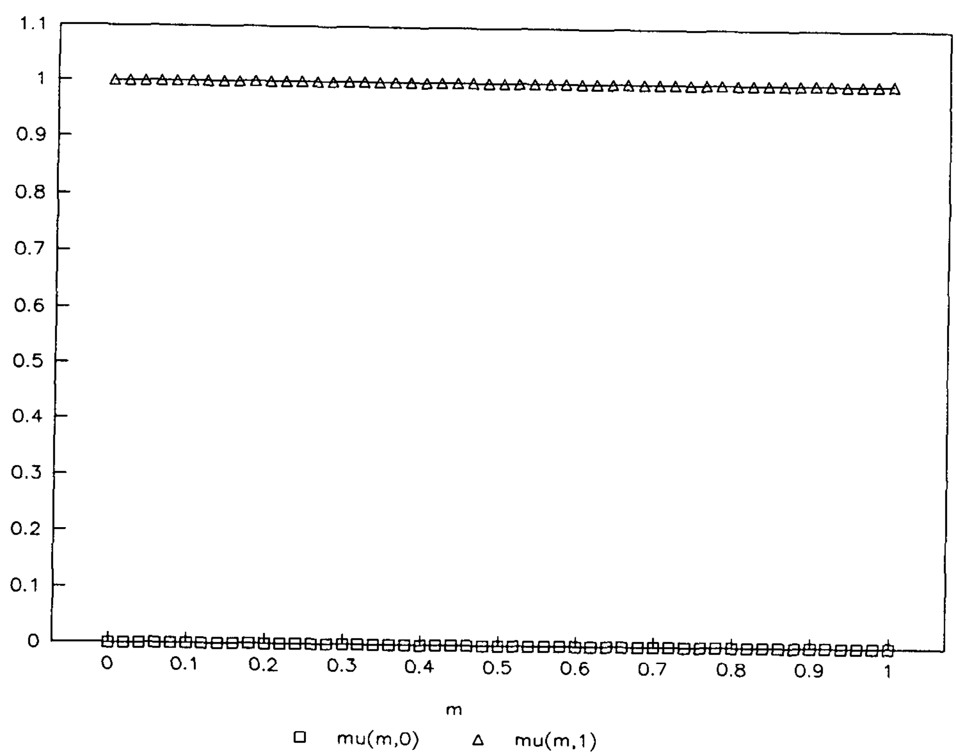
0.48
0.47
0.46
0.45
0.44
0.43
0.42
0.41
0.4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
m
c(m,0)
c(m,1)

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0.3
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0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
m
mu(m,0)
mu(m,1)